Name: August 5, 2011

Preliminary Doctoral Examination in Analysis

Work six of the following nine problems. If you work more than six then clearly indicate which of the six problems you would like to have graded. If you fail to indicate which six problems should be graded, then they will be chosen by the grader. Each problem is worth 10 points. You may work a seventh problem for a five-point bonus. If you do, please indicate clearly which is your bonus problem.

Please note that books, notes, or electronic communication devices are not allowed during the exam.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$$

for $A, B \in \mathcal{M}$.

2. Guess the limit of

$$\int_0^n e^{-2x} \left(\sum_{j=0}^{n} \frac{x^j}{j!} \right) dx \text{ as } n \to \infty$$

and then prove that your guess is correct.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and f be an extended real-valued function on X.

   (a) Define what it means for f to be measurable.

   (b) Show that f is measurable if and only if for each rational number q, $\{x \in X | f(x) > q\}$ is a measurable set.

4. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence of measurable functions on X that converges pointwise a.e. on X to a function f that is finite a.e. on X. Show that $X = \bigcup_{k=0}^{\infty} X_k$, where each X_k is measurable, $\mu(X_0) = 0$ and, for $k \geq 1$, $\{f_n\}$ converges uniformly to f on X_k.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. A sequence $\{f_n\}_{n \in \mathbb{N}}$ of measurable functions on X is said to *converge in measure* to a measurable function f provided for each $\epsilon > 0$,

$$\lim_{n \to \infty} \mu\{x \mid |f_n(x) - f(x)| \geq \epsilon\} = 0.$$

Prove that if $\mu(X) < \infty$ and $\{f_n\}$ converges pointwise a.e. to a function f, which is measurable and finite a.e., then $\{f_n\}$ converges to f in measure.

6. (a) Define what it means for a real-valued function f to be absolutely continuous on a closed bounded interval $[a, b]$.

   (b) Let $f : [0, 1] \to \mathbb{R}$ be continuous and increasing, and suppose f is absolutely continuous on $[\epsilon, 1]$ for every ϵ in $(0, 1)$. Show that f is absolutely continuous on $[0, 1]$.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and f be an extended real-valued function on X.

   (a) Define what it means for f to be integrable.

   (b) Suppose f is integrable and $\int_E f d\mu = 0$ for every measurable subset E of X. Show that $f = 0$ a.e. on X.

8. For any subset E of $\mathbb{R}$, define

$$\nu^\star(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ 1 & \text{if } E \subset [0, 1], E \neq \emptyset \\ \infty & \text{otherwise} \end{cases}.$$

Show that $\nu^\star$ is an outer measure and determine the σ-algebra of all $\nu^\star$-measurable subsets of $\mathbb{R}$.

9. (a) State Fubini's theorem.

   (b) Define the function $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1 & 0 \leq x - y \leq 1 \\ -1 & 0 < y - x \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

   Compute the iterated integrals $\int_0^\infty \left[\int_0^\infty f(x, y) dx\right] dy$ and $\int_0^\infty \left[\int_0^\infty f(x, y) dy\right] dx$ and comment how this relates to Fubini's theorem.