Preliminary examination in Partial Differential Equations

Student name: ______________________________

Instructions

1. **Work Problems 1 through 3 and at least one of Problems 4 through 6.**
2. No books or notes are allowed.
3. In all problems, you can apply the conclusions from the previous parts to all parts that appear later in the same problem.

Notations and Assumption

- $\mathbb{R}^n$: n-dimensional Euclidean space, $n \geq 2$, with points $x = (x_1, \ldots, x_n)$, $x_i \in \mathbb{R}$.
- U : proper open subset of $\mathbb{R}^n$; U is a *domain* if it is also connected.
- ∂U : boundary of U; $\bar{U} = U \cup \partial U$: closure of U.
- $D_i u = \dfrac{\partial u}{\partial x_i}$: partial derivative of u with respect to x_i , $D_{ij} u = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$, etc.
- Du : gradient of u.
- $\bar{u}$: average of function u over set U, i.e., $\bar{u} = \dfrac{1}{|U|} \displaystyle\int_U u$, where $|U|$ is the measure of U.
- In expressions $a^{ij}(x) D_{ij} u$ or $b^i(x) D_i u$, etc., the summation over indices $i, j = 1, \ldots, n$ is understood.
- The coefficient matrix a^{ij} that appears in the elliptic and parabolic operators is assumed to be uniformly positive definite, i.e., there exists $\theta > 0$ such that $a^{ij}(\cdot)\xi_i \xi_j \geq \theta |\xi|^2$ for all $\xi \in \mathbb{R}^n$ and throughout the domain of definition of a^{ij}.

Problems

1. (30 points) Assume U is a bounded domain with boundary of class C^1.
   (a) Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. Define the Sobolev spaces $W^{k,p}(U)$, $W_0^{k,p}(U)$, and the corresponding norms.
   (b) Suppose $u \in W^{1,p}(U)$ satisfies $Du = 0$ a.e. in U. Use smooth approximations to prove that u is constant a.e. in U.
   (c) State the embedding theorems for the space $W^{1,p}(U)$ for the cases $1 \leq p < n$ and $n < p < \infty$. Specify the value of the Sobolev exponent in the Sobolev inequality and the value of the Hölder exponent in Morrey's inequality.
   (d) Specify the conditions on p and q under which the embedding $W^{1,p}(U) \hookrightarrow L^q(U)$ is compact.

(e) Assume ∂U is of class C^1 and let $1 \le p < n$. Prove that the following Poincaré inequality

$$\|u - \bar{u}\|_{L^p(U)} \le C\|Du\|_{L^p(U)}$$

holds for some constant C that depends only on n, p, and U, and for all $u \in W^{1,p}(U)$. Conclude that, in particular, the norms $\|Du\|_{L^p(U)}$ and $\|u\|_{W^{1,p}(U)}$ are equivalent on $V = \{u \in W^{1,p}(U) : \ \bar{u} = 0\}$.
Hint: Assume that there exists a sequence $u_k \in W^{1,p}(U)$ such that

$$\|u_k - \bar{u}_k\|_{L^p(U)} > k\|Du_k\|_{L^p(U)}.$$

Use compactness to show that the sequence $v_k = \dfrac{u_k - \bar{u}_k}{\|u_k - \bar{u}_k\|_{L^p(U)}}$ converges in both $L^p(U)$ and $W^{1,p}(U)$. Identify the limits and arrive at a contradiction.

(f) Generalize the conclusion in part (e) slightly by showing that, for $1 \le p < n$, there exists a constant C depending only on n, p, and U such that the following Sobolev-Poincaré inequality holds

$$\|u - \bar{u}\|_{L^{np/(n-p)}(U)} \le C\|Du\|_{L^p(U)}$$

for all $u \in W^{1,p}(U)$. *Hint*: Apply the Sobolev inequality first.

2. (20 points) Suppose U is a bounded domain. Let L be an elliptic operator in non-divergence form with no zero-order term, i.e.,

$$Lu = -a^{ij}(x)D_{ij}u + b^i(x)D_iu.$$

Assume that the coefficients a^{ij} and b^i are continuous in U, and $a^{ij} = a^{ji}$ for i, $j = 1, \dots, n$.

(a) Give the definition of the *interior ball* condition on domain U.

(b) State Hopf's lemma and the strong maximum principle.

(c) Let u be a smooth solution of $Lu = 0$ in U. For $v = |Du|^2 + \lambda u^2$, show that $Lv \le 0$ in U when λ is large enough. Deduce

$$\|Du\|_{L^\infty(U)} \le C(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)})$$

for some constant C independent of u.

3. (30 points) Let $T > 0$ and define $U_T = U \times (0, T]$. Assume a^{ij}, b^i, $c \in L^\infty(U_T)$ for $i, j = 1, \dots, n$; $f \in L^2(U_T)$; and $g \in L^2(U)$. Suppose that $a^{ij} = a^{ji}$ for i, $j = 1, \dots, n$. For each time $0 < t < T$, define the operator L as

$$Lu = -D_j(a^{ij}(x,t)D_iu) + b^i(x,t)D_iu + c(x,t)u\ .$$

(a) Define what it means for u to be a *weak solution* of the following parabolic problem:

$$\begin{cases} u_t + Lu = f & \text{in } U_T \\ \quad\quad u = 0 & \text{on } \partial U \times [0,T] \\ \quad\quad u = g & \text{on } U \times \{t = 0\}\,. \end{cases} \qquad (*)$$

Specify the spaces for all functions appearing in your definition.

(b) Outline in general terms without going into much detail the Galerkin method for establishing the existence of a weak solution of problem $(*)$.

(c) State Gronwall's inequality.

(d) Show that a weak solution of $(*)$ is unique.

4. (20 points) Suppose u is a complex-valued function in $\mathbb{R}^n$ such that $u \in L^1(\mathbb{R}^n)$.

(a) Define the *Fourier transform* of u and the *inverse Fourier transform* of u.

(b) In general, $u \in L^2(\mathbb{R}^n)$ does not imply that $u \in L^1(\mathbb{R}^n)$. However, the Fourier transform of u is still defined as an element in $L^2(\mathbb{R}^n)$. Explain in what sense the Fourier transform of u is defined in this case and discuss other conclusions of Plancherel's theorem.

(c) Suppose $g \in L^2(\mathbb{R}^n)$. Derive an explicit formula for the solution of the following initial-value problem

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Hint: Apply the Fourier transform with respect to the space variable, solve the resulting ordinary differential equation, and apply the inverse transformation. Your representation will be in the form of convolution.

5. (20 points) Suppose U is a bounded domain. Let L be a second order elliptic partial differential operator in divergence form, i.e.,

$$Lu = -D_j(a^{ij}(x)D_i u) + b^i(x)D_i u + c(x)u \ .$$

Assume a^{ij}, b^i, $c \in L^\infty(U)$ and let $f \in L^2(U)$.

(a) Define what it means for u to be a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{$\dagger$}$$

(b) State the energy estimates for the bilinear form corresponding to the operator L.

(c) Give the statement of the Lax-Milgram theorem.

(d) Suppose $b^i = 0$, $i = 1, \ldots, n$. Prove that there exists a constant $\mu > 0$ such that, for each $f \in L^2(U)$, the boundary-value problem $(\dagger)$ has a unique weak solution $u \in H_0^1(U)$ provided $c(x) \geq -\mu$ for $x \in U$.

6. (20 points) Let $a^{ij} \in L^\infty(U)$ and define $Lu = -D_j(a^{ij}D_i u)$.

(a) State the John-Nirenberg inequality.

(b) Let $u \in W^{1,n}(U)$ and assume U is convex and bounded. Use the John-Nirenberg inequality to show that there exist positive constants σ_0 and C depending only on n such that

$$\int_U \exp\left\{ \frac{\sigma |u - u_U|}{\|Du\|_{L^n(U)}} \right\} dx \leq C d^n,$$

where $\sigma = \sigma_0 |U| d^{-n}$ and $d = \operatorname{diam}(U)$.

(c) State Harnack's inequality for the solutions of $Lu = 0$. Sketch the Moser iteration method for establishing this result in general terms without giving complete details. Point out the step at which the John-Nirenberg inequality is applied.

(d) Let B_r denote the ball of radius r centered at the origin. Suppose u is a weak solution of $Lu = 0$ in $U = B_1$. For $0 < r < 1$, let $M_r = \sup_{B_r} u$, $m_r = \inf_{B_r} u$, and define the oscillation of u in B_r as follows

$$\omega(r) = M_r - m_r.$$

Use the weak Harnack inequality to show that there exist positive constants α and C such that,

$$\omega(\rho) \leq C \left(\frac{\rho}{r}\right)^{\alpha} \omega(r),$$

for $0 < \rho < r < 1$.