Algebra Preliminary Exam 2012

May 12, 2012

1 Group Theory

Instruction: Do problems 1 and 2 and any three of the remaining five.

1. State and prove the Second Isomorphism Theorem (also known as the Diamond Theorem).
2. State and prove the First Sylow Theorem (also known as the Sylow Existence Theorem).
3. (a) Prove that a subgroup of a cyclic group is cyclic.
   (b) Prove or disprove: The group $\mathbf{Q}$ of rational numbers under addition is cyclic.
   (c) Let G be a group. Prove that if $G/Z(G)$ is cyclic, then G is abelian. ($Z(G)$ denotes the center of G.)
4. (a) Let m and n be positive integers. Prove that if $\gcd(m,n) = 1$, then $\mathbf{Z}_m \oplus \mathbf{Z}_n \cong \mathbf{Z}_{mn}$. ($\mathbf{Z}_n$ denotes the group of integers modulo n.)
   (b) Give a list of groups having the property that every abelian group of order 120 is isomorphic to precisely one of the groups in the list.
   (c) To which group in your list is the group $\mathbf{Z}_2 \oplus \mathbf{Z}_{10} \oplus \mathbf{Z}_6$ isomorphic?
5. (a) Let $\mathbf{C}$ be a category. Define **coproduct** of the family $\{A_i\}_{i\in I}$ of objects of $\mathbf{C}$.
   (b) Discuss uniqueness of coproducts and prove your claims.
   (c) Prove that in the category of abelian groups every family of objects has a coproduct.
6. (a) Define **nilpotent group** (and include the definitions of any nonelementary terms you use).
   (b) Prove that a finite p-group is nilpotent (p, prime).
   (c) Let D_4 be the dihedral group of order 8 and let S_3 be the symmetric group on three elements. Prove or disprove: $D_4 \times S_3$ is nilpotent.
7. (a) Let G be a group and let N be a normal subgroup of G. Prove that if N is a maximal normal subgroup of G, then G/N is simple.
   (b) Give an example of two nonisomorphic groups having the same composition factors, with one of these composition factors having order 60. Prove that the groups are not isomorphic.
   (c) Prove that there does not exist a simple group of order 280.

2 Rings, Modules, and Galois Theory

Instruction: Do problems 8 and 9 and any three of the remaining five.

8. State the structure theorem of finitely generated modules over a principal ideal domain. You need not prove the theorem.

9. Draw the intermediate field diagram between $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$ and $\mathbb{Q}$, and the corresponding subgroup diagram of the Galois group $\text{Aut}_{\mathbb{Q}}\mathbb{Q}(\sqrt{-1}, \sqrt{2})$.

10. Compute the Galois group of $x^3 - 3x + 3 \in K[x]$ over the following fields K (Hint: $x^3 - 3x + 3$ may or may not be irreducible in different $K[x]$):
    (a) $K = \mathbb{Q}$
    (b) $K = \mathbb{Z}_2$
    (c) $K = \mathbb{Z}_5$
    (d) $K = \mathbb{Z}_7$

11. (a) Let R be a commutative ring with identity. Let $f : R \to S$ be a ring epimorphism. Prove that S is an integral domain if and only if $\ker f$ is a prime ideal of R.
    (b) Prove that any homomorphic image of a Noetherian ring is a Noetherian ring.

12. Prove that if F is an extension field of K and $u \in F$ is algebraic over K, then $K(u) = K[u]$.

13. Prove that an integral domain R is a unique factorization domain if and only if every nonzero prime ideal in R contains a nonzero principal ideal that is prime.

14. Let R be an integral domain. Let A be a unitary torsion R-module. In other words, for every $a \in A$, $1_R \cdot a = a$ and there is a nonzero element $r_a \in R \backslash \{0\}$ such that $r_a \cdot a = 0$. Prove that A is not a projective R-module.