Complex Analysis PRELIMINARY EXAMINATION January 15, 2011
Time: 3:00-5:30 p.m.
Note: Please solve any eight of the following problems.

1. (a) For any two complex numbers z_1 and z_2, prove that $|z_1 + z_2| \leq |z_1| + |z_2|$, and derive the necessary and sufficient conditions under which the above inequality becomes equality, assuming both z_1 and z_2 are non zero.

   (b) Show that if z and z' correspond to diametrically opposite points on the Riemann sphere then $z\bar{z'} = -1$.

2. State and prove Gauss-Lucas Theorem concerning the zeros of a polynomial and its derivative.

3. (a) Prove that the truth of Cauchy Riemann equations is a necessary condition for a function to be differentiable, and **state** the conditions on the function so that this condition becomes a necessary and sufficient for a function to be differentiable at a point.

   (b) Let a function f be analytic in a region Ω. Show that if $\bar{f}$ is also analytic in Ω, then f must be a constant.

4. Let $f(z)$ be analytic inside and on a rectangle R. Then prove that

$$\int_{\partial R} f(z)\, dz = 0,$$

   where ∂R denotes the boundary of R.

5. (a) Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and let $R = \dfrac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$. Prove that for every z with $|z| < R$, the series converges absolutely.

   (b) State Cauchy's Integral Formula and use it to evaluate the integral $\displaystyle\int_{|z|=2} \frac{\sin z}{(z - \pi/4)^3}\, dz$.

6. (a) State and prove Liouville's theorem.

   (b) State Open Mapping Theorem and use it to prove the Maximum Modulus Principle.

7. Use residue theorem to evaluate the integrals

   (a) $\displaystyle\int_0^{\infty} \frac{x \sin x}{x^2 + 4}\, dx$

   (b) $\displaystyle\int_0^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 4)}$

8. Use Argument Principle to prove Rouché's Theorem. Then use Rouché's Theorem to prove Fundamental Theorem of Algebra.

9. State Schwarz's lemma and use it to prove that if $f(z)$ is analytic for $|z| \leq 1$, $|f(z)| \leq M$ for $|z| = 1$ and $f(a) = 0$ where $|a| < 1$, then

$$|f(z)| \leq M \left| \frac{z - a}{\bar{a}z - 1} \right|.$$