Prelim: Linear Algebra and Matrix Theory

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August, 2013

Note: I have substituted $\mathbb{F}^{m\times n}$ for the notation $\mathrm{M}_{m,n}(\mathbb{F})$ that I used in class. Both denote the set of all $m\times n$ matrices with entries from $\mathbb{F}$.

1. Consider the vector $(a_1, a_2, \ldots, a_n)^t$ where each $a_i \in \mathbb{C}$. Let $V(a_1, a_2, \ldots, a_n)$ denote the $n \times n$ matrix, known as a Valdermonde matrix, whose (i, j)-th term is a_j^{i-1} for each i and j such that $1 \le i, j \le n$. Prove that

$$\det(V(a_1, a_2, \ldots, a_n)) = (-1)^n \prod_{i<j}(a_j - a_i) \quad \forall\, a_1, a_2, \ldots, a_n \in \mathbb{C}.$$

Now consider the $n \times n$ matrix $W(a_1, a_2, \ldots, a_n)$ such that for each i, $1\le i\le n-1$, and j, $1 \le j \le n$, the (i, j)-th term is a_j^{i-1}, while the (n, j)-th term is a_j^n for each j. For example,

$$W(x, y, z) = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{bmatrix}.$$

Prove that

$$\det(W(a_1, a_2, \ldots, a_n)) = (-1)^n \prod_{i<j}(a_i - a_j)(a_1 + a_2 + \cdots + a_n).$$

2. Suppose $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$. The tensor produce of A and B, denoted by $A \otimes B$, is the $mn \times mn$ matrix whose (i, j)-th $n \times n$ block is $a_{ij}B$, where $A = [a_{ij}]$. For example, if $A = [a_{ij}]$ is a 2×2 matrix, then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}.$$

Prove each of the following in the general case (arbitrary m, and n). Assume that $A, C \in \mathbb{C}^{m\times m}$ and $B, D \in \mathbb{C}^{n\times n}$.

(a.) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,

(b.) $\det(A \otimes B) = (\det(A))^n(\det(B))^m$.

3. Suppose V is a complex inner product space of dimension m, and $m > 0$. Let U and W be subspaces of V, and assume that $w_1, w_2, \ldots, w_p$ span $W^{\perp}$, while $\{u_1, u_2, \ldots, u_t\}$ is a basis for U. Let M denote that $p \times t$ matrix whose (i,j)-th term is $\langle w_i, u_j \rangle$ where $\langle \cdot, \cdot \rangle$ is the inner product on V. Prove the following.
   (a.) $\dim(U \cap W) = \dim(\ker(M))$.
   (b.) $\dim(U \cap W) = t - \operatorname{rank}(M)$.

4. Let $p(x) = x^5 - 1$. Suppose D is an $n \times n$ real symmetric matrix such that $P(D) = 0$. Must it be true that $D = I$? Either provide proof, or disprove via counterexample.

5. Give an example of a pair of $n \times n$ non-zero real matrices that have the same minimal polynomial but are not similar. Prove that your example is correct.

6. Characterize all complex unitary matrices whose minimal and characteristic polynomials are the same.

7. Suppose $\mathcal{O}$ is an $n \times n$ real orthogonal matrix. This means $\mathcal{O} \in \mathbb{R}^{n \times n}$ and that $\mathcal{O}^t \mathcal{O} = \mathcal{O} \mathcal{O}^t = I$. Prove that there exists a real orthogonal matrix Q such that $Q^t \mathcal{O} Q$ is a direct sum of 1×1 and 2×2 orthogonal matrices.

8. Suppose $A, B \in \mathbb{C}^{n \times n}$, and both A and B are diagonalizable. Prove that AB is diagonalizable if and only if A and B commute.

9. Suppose $A, B \in \mathbb{C}^{n \times n}$. What is the relationship between the characteristic polynomial of AB and the characteristic polynomial of BA. State and prove the appropriate theorm.

10. Suppose $A \in \mathbb{C}^{n \times n}$ and A is Hermitian and positive semi-definite. Use Schur's lemma to prove that there exists $X \in \mathbb{C}^{n \times n}$ such that $A = X^* X$.

11. Suppose $A, B \in \mathbb{C}^{n \times n}$ and both A and B are Hermitian and positive semi-definite. Show that each eigenvalue of AB is real and non-negative. Does this generalize to three matrices A, B, and C? In other words, if $A, B, C \in \mathbb{C}^{n \times n}$ are Hermitian and positive semi-definite, then is it true that each eigenvalue of ABC is real and non-negative? Prove or give a counterexample. Hint: Consider the case where C is a positive diagonal matrix.

12. What do you know about $n \times n$ real matrices with positive entries? Give the most significant facts, particularly with respect to eigenvalues, eigenvectors, spectral radius etc. Which results extend to non-negative matrices?

13. Suppose $A \in \mathbb{C}^{n\times n}$, and let

$$B = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}.$$

(a) Show that $\det(B) = (-1)^n |\det(A)|^2$. (b.) Find the eigenvalues of M in terms of the eigenvalues of A. (c.) Why can M not be positive semi-definite?

14. Suppose $A \in \mathbb{C}^{n\times n}$ and A is normal. (a.) Prove that a number $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $\bar{\lambda}$ is an eigenvalue of A^*. (b.) Show that the eigenspace of A associated with eigenvalue λ is the same as the eigenspace of A^* associated with eigenvalue $\bar{\lambda}$. (c.) Prove that if λ and μ are distinct eigenvalues of A, then the corresponding eigenspaces are orthogonal with respect to standard inner product on $\mathbb{C}^n$. (d.) Using the above prove that if A is normal, then there exists a unitary matrix U such that U^*AU is a diagonal matrix. (e.) Is the converse true?

15. Suppose $\mathcal{F} \subset \mathbb{C}^{n\times n}$. Under what conditions is it true that there exists a single unitary matrix U such that U^*AU is upper triangular for all $A \in \mathcal{F}$? State and prove a theorem that gives sufficient conditions under which members of $\mathcal{F}$ are simultaneously unitarily upper triangularizable.

16. Carefully state the Cauchy interlacing theorem for Hermitian matrices.

17. Suppose $D \in \mathbb{R}^{n\times n}$, and $D = [d_{ij}]$ has non-negative entries. (a.) Show that if each row of D sums to r, where r is a positive real number, then the spectral radius of D is r. (b.) Suspend the assumption that the rows of D sum to the same number and show that the spectral radius of D, $\rho(D)$, is not less than the minimum row sum. That is, show that

$$\rho(D) \geq \min_{1\leq i\leq n} \left\{ \sum_{j=1}^{n} d_{ij} \right\}.$$

Hint: Let r_i denote the i-th row sum of D and consider the matrix $\tilde{D} = \lambda D$ where λ is the diagonal matrix whose diagonal entries are the reciprocals of the numbers r_i. The case where some $r_i = 0$ must be considered separately.

What do you know about the spectral radius of $\tilde{D}$. Let r denote the minimum of the r_i, and let $\hat{D}$ denote $(1/r)D$. What is the relationship between $\tilde{D}$ and $\hat{D}$?