STAT 7600/7610 Mathematics Statistics Preliminary Exam
August 10, 2016

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Name:

1. It is a closed-book in-class exam.
2. Calculator is allowed.
3. Show your work to receive full credits. Highlight your final answer.
4. Solve any five problems out of eight.
5. Total points are 50 with 10 points for each problem.
6. Time: 180 minutes. (8:00am - 11:00am, August 10th, 2016)

1	2	3	4	5	6	7	8	Total

1. (a) Suppose that T_n has a χ^2 distribution of n degrees of freedom. Show that

$$\sqrt{2T_n} - \sqrt{2n}$$

converges in distribution to $N(0,1)$.

(b) Let Y_1 and Y_2 be iid $uniform(0,1)$ random variables. Define

$$X_1 = cos(2\pi Y_2)\sqrt{-2log(Y_1)} \text{ and } X_2 = sin(2\pi Y_2)\sqrt{-2log(Y_1)}$$

Derive the joint distribution of X_1 and X_2. Name the marginal distributions of X_1 and X_2.

2. Suppose that $X_1, X_2, \ldots, X_n$ are iid Poisson(λ).

   (a) Find the best unbiased estimators of $e^{-\lambda}$ and $\lambda e^{-\lambda}$.

   (b) For the best unbiased estimator of $\lambda e^{-\lambda}$ in part (a), calculate the asymptotic relative efficiency with MLE of λ. Which estimator do you prefer?

3. Assume that a random variable $Y = ln(X) \sim N(\mu, \sigma^2)$, so that X has a lognormal distribution. Let $X_1, \dots, X_n$ be iid random variables of the lognormal distribution for X, and suppose that we are interested in the maximum likelihood estimator (MLE) $\hat{\xi}$ of $\xi = E(X)$.

   (a) Find an explicit expression for the MLE $\hat{\xi}$ of the parameter ξ.

   (b) Show that MLE $\hat{\xi}$ is asymptotically unbiased estimator of the parameter ξ.

4. Let $X_1, \ldots, X_n$ be iid random variables of Poisson($\sqrt{\lambda}$) where $\lambda > 0$.

   (a) Construct the uniformly most powerful (UMP) level α test of $H_0 : \lambda_0 = 1$ vs $H_1 : \lambda_0 > 1$.

   (b) If $n = 1$ and $\alpha = 0.05$, what is the reject region of the test?

   (c) If $\hat{\lambda}_n$ is the MLE for λ, find the limiting distribution of $\sqrt{n}(\hat{\lambda}_n - \lambda)$.

5. Let $X_1, \ldots, X_n$ be a random sample from a distribution with the joint pdf

$$f(x \mid \theta) = \frac{1}{2\theta^3} e^{-x/\theta} x^2, \qquad x > 0, \quad \theta > 0.$$

(a) Let $T = \sum_{i=1}^n X_i$ be a statistic for θ. Show that T has the monotone likelihood ratio property.

(b) Find the UMP test procedure for testing $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ at the level $\alpha = 0.05$.

(c) Compute the power function of the test procedure obtained in (b).

6. Let $X_1, \ldots, X_n$ be a random sample from

$$f(x \mid \theta) = \frac{log(\theta)}{\theta - 1}\theta^x, \quad 0 < x < 1, \quad \theta > 1$$

(a) Find a function $\phi = \phi(\theta)$ of θ such that there is an unbiased estimator $\hat{\phi}$ of ϕ with variance $Var(\hat{\phi})$ achieving the Cramér-Rao lower bound. Obtain $\hat{\phi}$.

(b) Is the estimator $\hat{\phi}$ obtained in (a) above the UMVUE (Uniformly Minimum Variance Unbiased estimator) of ϕ? Why?

7. Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\theta, \theta^2)$, $\theta > 0$. We know that $\bar{X}$, the sample mean, and cS ($c > 0$), where $S = (\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2)^{1/2}$, are both unbiased estimators of θ.

   (a) How do you find c? Explain.

   (b) Prove that
   $$k\bar{X} + (1-k)(cS)$$
   is an unbiased estimator of θ for any number k.

   (c) What is the value of k that produces the estimator with a minimum variance?

   (d) Show that $(\bar{X}, S^2)$ a sufficient statistic for θ, but it is not a complete sufficient statistic.

8. Suppose that $X_1, \ldots, X_n$ are iid $N(\theta, 1)$ random variables with $\theta \in R$.

(a) Find the Fisher information $I_n(\theta)$.

(b) Define

$$\hat{\theta}_n = \begin{cases} \bar{X}, & |\bar{X}| \geq n^{-1/4} \\ t\bar{X}, & |\bar{X}| < n^{-1/4} \end{cases}$$

where t is a fixed constant. Show that

$$[V_n(\theta)]^{-1/2}(\hat{\theta}_n - \theta) \longrightarrow^d N(0,1)$$

where $V_n(\theta) = V(\theta)/n$, and $V(\theta) = 1$ if $\theta \neq 0$ and $V(\theta) = t^2$ if $\theta = 0$.