

GLOBAL SENSITIVITY OF NEOCLASSICAL AND FACTOR PROPORTIONS MODELS TO PRODUCTION TECHNOLOGY

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This paper pictures the global influence of various patterns of returns to scale in the general equilibrium model of production with two factors and two goods. Constant, increasing, and decreasing returns to scale at the sectoral level are explicitly specified. The changing slope and curvature of contract curves, isoquants, production frontiers, and relative price lines are examined under various Cobb-Douglas and exponential production functions. Fundamental properties of neoclassical and factor proportions theory are thus illustrated. [D50, F11]

1. INTRODUCTION

This paper explores the global sensitivity of contract curves, isoquants, production possibility frontiers, and relative price lines to various returns to scale. These curves are an important part of the general equilibrium economics of production, and their slope and curvature reflect the direction and strength of global adjustment in outputs and factor prices to underlying changes in the prices of outputs.

Qualitative comparative static analysis, exemplified by Jones (1968), is concerned with the direction of local adjustment around an equilibrium. Global properties of these important theoretical models, however, remain largely unexplored. The range and responsiveness of outputs and factor prices to exogenous changes in the prices of goods are potentially as important as the direction of local changes.

Samuelson (1971) argues that the production possibility frontier is less concave in the specific factors model than usually perceived. This paper shows the frontier to be nearly linear in the two factor model when there are constant returns to scale and factor intensities are not widely different across goods. The implication is that outputs are extremely sensitive to price changes. In a small open economy, changes in world prices or the level of protection would thus have large output effects.

Variable returns to scale in the factor proportions model of production and trade

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have been explored by Kemp (1964), Minabe (1966), Jones (1968), Herberg and Kemp (1969), Panagariya (1980, 1983), and others, and the vast literature is reviewed by Helpman (1984). The focus has been on the local conditions under which the Stolper-Samuelson, Rybczynski, and factor price equalization theorems hold. The present paper pictures the related global properties of the two factor, two good model with various combinations of returns to scale across sectors.

This paper utilizes the general equilibrium model of production with two factors and two goods. Each sector is made up of numerous identical firms. Returns to scale are constant at the firm level, and economies or diseconomies are introduced by a scale effect. The model allows entry and exit, but does not solve for the number of firms. Cobb-Douglas and exponential production functions are specified. Models with various degrees of constant, increasing, and decreasing returns to scale in either sector are examined. Changes in the technological parameters of the exponential production functions lead to pictured shifts in the slope and curvature of the various curves.

Convex production possibility frontiers due to increasing returns to scale are derived. Increasing returns in one sector in combination with decreasing returns in the other sector lead to frontiers which are partly convex and partly concave, giving rise to a potential dual general equilibrium in which the relative price line, which maps relative prices to relative factor prices, is not a singular mapping.

Elasticities of relative factor prices and output substitution with respect to relative prices of goods are also reported. There is a good deal of variation across technologies in the global responsiveness of factor prices and outputs to exogenous changes in the prices of goods.

2. METHOD OF SOLUTION

The exponential production function in either sector is specified as

$$x_j = (1 - \beta_1^j) (1 - \beta_2^j) \tag{1}$$

$$x_2 = \beta_2^j v_{22} \tag{2}$$

Outputs are $x_j, j = 1,2$. The total input of factor $i (i = 1,2)$ in the production of good j is written v_{ij} . Note that the total endowment of each factor has been standardized to 1: $v_{i1} + v_{i2} = 1, i = 1,2$.

Technological parameters are given by the transposed technology vector $t = (\quad)$ and are chosen to depict various degrees of constant, increasing, and decreasing returns to scale. If $\beta_1 + \beta_2$ or $\beta_1 + \beta_2$ equals one, that sector's production function is Cobb-Douglas, constant returns to scale (CRS).

At the level of the firm, CRS is maintained in order to keep the firm's profit function bounded. A scale effect $g_j(x_j)$ is introduced as in Kemp (1964) to create

variable returns at the industry level,

$$x_j = n_j v_{2j}^b = n_j g_j(x_j) \frac{c}{i_j} \frac{d}{2j}, \tag{3}$$

where n_j is the number of firms producing good j , v_{ij} is the input of factor i in one of the n_j identical firms, $n_j v_{ij} = v_{ij}$, $a + b = 1$, and $c + d = 1$. Economies and diseconomies are external to the firm but internal to the industry, as discussed by Kemp and Negishi (1970).

It is curious that the scale or externality effect in (3) is

$$g_j(x_j) = \frac{a-c}{n_j} v_{2j}^{b-d}, \tag{4}$$

an exponential function itself. It follows that

$$dg_j / dx_j = \frac{d}{dx_j} (dg_j / dv_{ij}) / (dx_j / dv_{ij}), \tag{5}$$

which must have the same sign as $a(b - d) + b(a - c)$.

This model does not solve for the number of firms in sector j , n_j . Variable returns are thus placed on the same theoretical footing as constant returns. The number of identical firms, n_j , is neutral in its effect on the general equilibrium. A different n_j would imply a different level of inputs v_{ij} , consistent with the same total sectoral output, for the typical firm. With free entry and exit, the number of firms n_j and inputs at the level of the firm v_{ij} are constrained to adjust so profit equals zero.

A specified production function in (1) and (2) leads to a corresponding set of isoquants, the contract curve (CC), production possibility frontier (PPF), and relative price line (RPL).

Optimal output can be derived by maximizing the Lagrangian function:

$$L = v_2 v_{22} + [x_1 - (1 - \lambda) (1 - \lambda)] \tag{6}$$

First order conditions are

$$L / v_{12} = v_{12}^{-1} v_{22} + (1 - \lambda)^{-1} (1 - \lambda) = \dots \tag{7}$$

$$L/v_{22} = v_{12}v_{22}^{-1} + (1 - \beta)(1 - 2\beta)^{-1} = \quad (8)$$

$$L/v_{22} = \beta - (1 - \beta)(1 - 2\beta) = 0. \quad (9)$$

Because of the exponents, there is no closed solution. Instead, v_{22} is discretely varied between zero and one to derive plots of the various curves.

From (7),

$$= r/p_2 = (x_2/v_{12})/[x_1/(1 - \beta)] \quad (10)$$

The Lagrangian multiplier is thus interpreted as dx_2/dx_1 , the slope of the PPF. For notation, $p = p_1/p_2 = -$. From (8) and (10),

$$v_{12} = (v_{22})/[v_{22} + (1 - 2\beta)] \quad (11)$$

The optimal v_{12} can thus be found for any v_{22} . Then, from (9) and (11),

$$x_1 = (1 - \beta) \{ (1 - \beta) \} / [(1 - 2\beta) + v_{22}] . \quad (12)$$

Finally, x_2 is calculated from (2).

First order conditions thus lead to expressions for the optimal x_1 in (12) and x_2 in (2) for arbitrary v_{22} . The marginal rate of technical substitution (MRTS) along the isoquant is the ratio of marginal products, $(x_1/v_{11})/(x_2/v_{21}) = v_{21}/v_{11}$. In the cost minimization, $MRTS = w_1/w_2$. The relative factor price is derived as

$$w = w_1/w_2 = v_{22}/v_{12}. \quad (13)$$

For a given set of technological parameters, the various curves are plotted through

variation in v_{22} . Isoquants can also be generated, but are not pictured due to space constraints. For consistency, the production of good 1 is kept intensive in factor 1 in every specification.

3. MODELS WITH CONSTANT RETURNS TO SCALE

With technology $t = (.5 \ .5 \ .5 \ .5)$, the CC and PPF would be linear with $w = 1$. If the two goods are identical in their production functions, the CC and PPF would be linear and w would be uniquely determined. Consider the technology $t_1 = (.8 \ .2 \ .2 \ .8)$. Table 1 illustrates the variational technique used to plot the various curves.

Table 1. Example of Plots with t_1

v_{22}	v_{12}	x_2	x_1	p	w
0.000	0.000	0.000	1.000		
0.100	0.007	0.059	0.974	2.166	3.625
0.200	0.015	0.120	0.945	2.028	3.250
0.300	0.026	0.184	0.912	1.884	2.875
0.400	0.040	0.252	0.874	1.733	2.500
0.500	0.059	0.326	0.829	1.572	2.125
0.600	0.086	0.407	0.775	1.399	1.750
0.700	0.127	0.498	0.705	1.211	1.375
0.800	0.200	0.606	0.606	1.000	1.000
0.900	0.360	0.749	0.442	0.754	0.625
1.000	1.000	1.000	0.000		

With t_1 , note that the CC in Figure 1 is quite convex, while the PPF in Figure 2 is much less concave. PPFs are only slightly concave under CRS, as reflected by the very limited domain of prices for the RPL in Figure 3 consistent with diversified production. If complete specialization is reached, factor prices would not adjust to further changes in the prices of goods. The RPLs in Figure 3, in other words, could be extended horizontally from either end as in Kemp (1964).

If an economy with technology t_1 moves from autarky to free trade in Figure 2, an industry would likely shut down as prices of goods adjust to international levels. Imagine an economy made up of many such industries, all with CRS and with this degree of variation across production functions. Exogenous changes in prices of goods brought about by a change in world prices or in the level of protection would have very small effects on factor prices but very large effects on outputs. Tariffs would have overwhelming effects on outputs and would likely cause unprotected sectors to shut down.

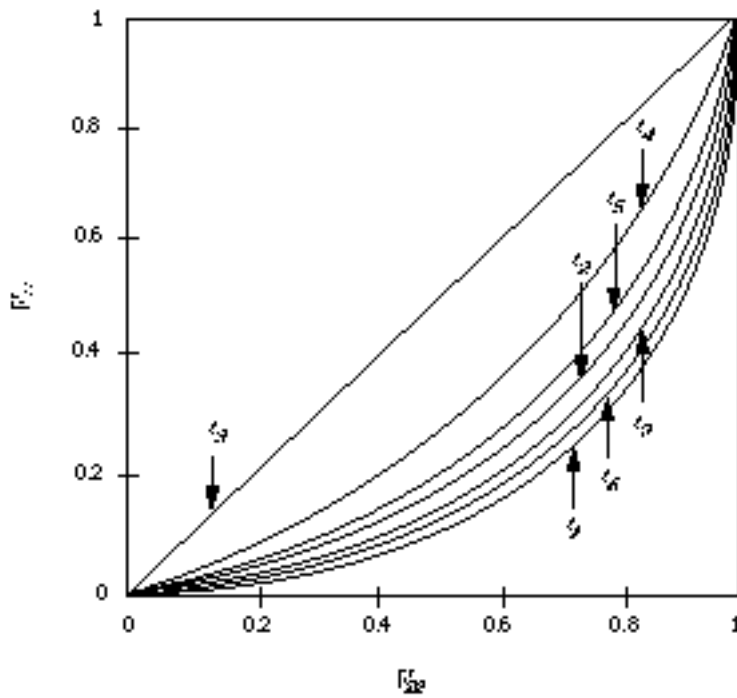


Figure 1. Contract Curves

PROPOSITION 1. *With constant returns to scale, output is extremely sensitive to price changes. Complete specialization and industrial shutdown are likely outcomes of price changes.*

When the symmetric difference across production functions is decreased the PPF becomes less concave and the range of the RPL is slightly decreased as the PPF becomes more elastic. In contrast, the CC becomes less convex.

PROPOSITION 2. *With constant returns to scale and different technologies across sectors, the degree of convexity of the contract curve is much greater than the degree of concavity of the production possibility frontier.*

The difference across goods is also reduced by making good 2 less intensive in factor 2. The CC then returns to a less concave shape. The RPL shifts back to the left, becomes steeper, and has a wider range than with the previous technologies. For most prices, this shift favors factor 1. Similar PPFs may disguise different underlying

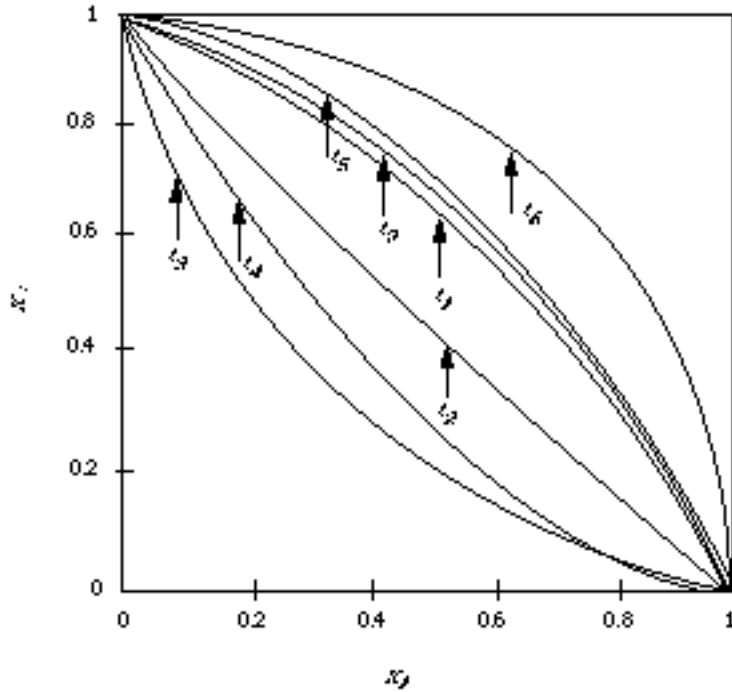


Figure 2. Production Possibility Frontiers

factor price patterns.

PROPOSITION 3. *The shape of the production possibility frontier does not necessarily indicate the shape of the underlying relative price line.*

In a situation of CRS with both sectors biased toward factor 1, the PPF is nearly linear, straighter even than with t_j . Isoquants in both sectors are fairly flat. The relative price of factor 1 is high, and the domain of price changes in the RPL is severely limited.

PROPOSITION 4. *With constant returns to scale in both sectors, larger differences in technology across goods cause the contract curve to become more convex, the production possibility frontier to become more concave, and the positively sloped relative price line to flatten, widen in range, and become less convex.*

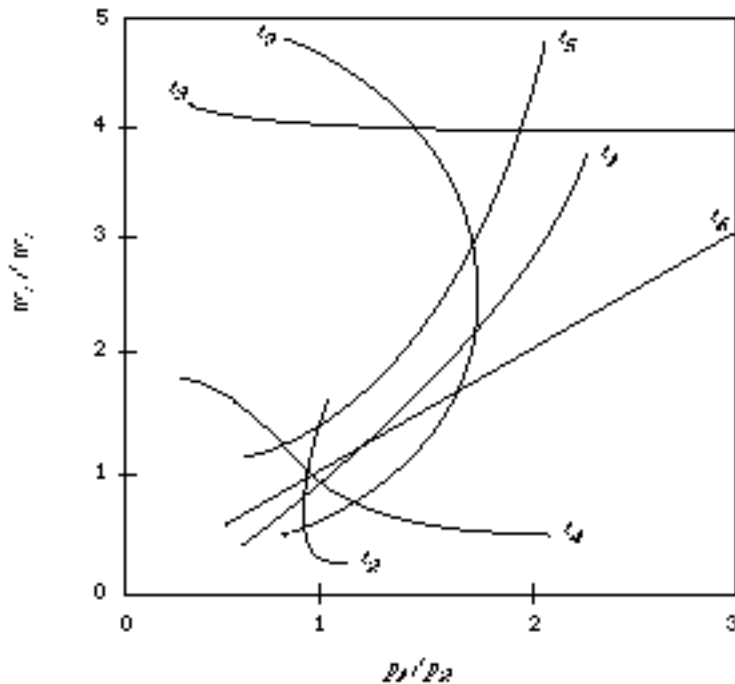


Figure 3. Relative Price Lines

4. MODELS WITH INCREASING RETURNS TO SCALE

When there is increasing returns to scale (IRS) in one sector with CRS in the other sector, the PPF is typically convex. An increase in the relative price of a good thus leads to a decrease in its output. Each good's long run supply curve slopes downward in the general equilibrium.

With $t = (.9 .5 .5 .5)$, there is a negatively sloped RPL (not pictured) which means that an increase in the relative price of a good decreases the price of the factor used intensively in its production. The PPF becomes less convex toward higher output of the IRS industry. The RPL becomes steeper and more convex as the relative price of the IRS good falls.

IRS in sector 1 is maintained and production of good 2 is made more intensive in factor 2 under $t_2 = (.9 .5 .2 .8)$. The CC is more convex than with $t = (.9 .5 .5 .5)$. The PPF is concave toward the axis of the industry with IRS and convex toward the CRS axis. Effects of differences in factor intensity overcome effects of IRS in the concave

region, as discussed by Kemp (1964).

PROPOSITION 5. *A higher degree of increasing returns to scale causes the contract curve to become more convex and the production possibility frontier to become more concave. The negatively sloped concave relative price line widens in range and becomes less concave.*

A dual equilibria is possible with t_2 . If $p = 1$, for instance, there are two different equilibrium points on each of the PPF, CC, and RPL. Stability analysis would introduce demand and examine stability of the equilibrium on the convex portion of the PPF, corresponding to the negatively sloped portion of the RPL. In a pure production model of a small open economy with exogenous prices, the dual equilibria is possible. At the convex equilibrium, relatively little of the good with IRS is produced. In this convexity “trap,” a higher relative price for the IRS good lowers the relative return to factor 1 and pushes both factors into the large CRS sector.

PROPOSITION 6. *Increasing returns to scale in one sector cause a convex production possibility frontier unless the factor intensity difference is large.*

Both sectors use a similar high ratio of factor 1 to factor 2 and the factor intensity difference between sectors is reduced with $t_3 = (2.1 \ .5 \ .8 \ .2)$. With similar input ratios, the CC is nearly linear and the isoquants both biased toward factor 1 and only slightly convex. The PPF is extremely convex toward the axis of the CRS industry. The RPL is nearly linear, indicating a relatively high return to factor 1 across a wide domain of prices.

In a model with IRS in both sectors and production in both sectors biased toward factor 1, the CC is straight, the PPF convex, and the negatively sloped RPL almost straight. If the IRS production technologies were identical across sectors, the CC and PPF would be linear. A large intensity difference across sectors is pictured with $t_4 = (.9 \ .5 \ .5 \ .8)$. The result is a moderately convex CC, while the PPF remains convex. Similarity of the PPFs occurs with IRS in both sectors, hiding underlying differences in the CCs and RPLs as the degree of IRS varies.

5. MODELS WITH DECREASING RETURNS TO SCALE

Consider now the effects of DRS in one sector. With $t_5 = (.5 \ .1 \ .5 \ .5)$, the CC and PPF look much like those typically drawn and the RPL extends over a fairly wide range of relative prices. The PPF becomes less concave as output in the CRS industry increases. Compare t_5 with t_2 to see the effects of a move toward DRS in one sector.

PROPOSITION 7. *As returns to scale fall in the production of one good, the contract*

curve becomes more convex, the production possibility frontier becomes more concave, and the positively sloped relative price line becomes steeper, wider in range, and less convex.

The factor intensity difference is enlarged by making production in sector 2 more intensive in factor 2. The CC then becomes more convex and the PPF more concave. The domain of the RPL widens as the RPL flattens and becomes less convex.

In a situation with both sectors biased toward factor 1 and a small factor intensity difference, the CC is nearly straight. A high return to factor 1 is suggested by flat isoquants. The PPF is less concave than with t_5 . The RPL also indicates a very high relative return to factor 1.

A large factor intensity difference across sectors with DRS in both sectors is pictured with $t_6 = (.5 \ .1 \ .2 \ .5)$. Contrast t_6 with t_5 which has DRS in only one sector. The CC has become more convex, and the PPF more concave. The RPL has widened its domain, switched from convex to concave, and shifted to indicate a higher relative return to factor 2.

With $t = (.5 \ .1 \ .5 \ .2)$, the increased intensity of factor 1 in the production of good 2 would cause the RPL to rise and become concave. The CC would be less convex, while the PPF remains roughly the same. Again, if the DRS technology were identical across sectors, the CC and PPF would be linear.

6. DECREASING RETURNS MIXED WITH INCREASING RETURNS

The last technology reports a model with DRS in one sector and IRS in the other. A large difference in factor intensity is introduced with $t_7 = (.5 \ .1 \ .5 \ .9)$.

The PPF is convex toward the axis of the IRS industry, but remains concave toward the axis of the DRS industry. As the difference between sectors increases, the PPF and RPL exhibit greater curvature. The RPL bends back on itself. A given relative price p of the two goods may thus be consistent with two relative factor prices w and two points on each of the CC, PPF, and RPL.

This potential dual equilibrium means that first order conditions alone do not guarantee an optimal outcome. Projecting to models with many goods, the lesson is that a small open economy facing exogenous prices may end up with different sets of consistent outputs and factor prices. Global analysis becomes especially relevant. In a closed economy with domestic demand, imposing "stability conditions" rules out convex portions of the PPF.

PROPOSITION 8. *With decreasing returns to scale in one sector and increasing returns to scale in the other, the production possibility frontier is concave toward higher output in the decreasing returns sector and convex toward higher output in the increasing returns sector. The relative price line bends back on itself, and the mapping from the prices of goods to factor prices is not univalent.*

7. FACTOR PRICE AND OUTPUT ELASTICITIES

This section specifies general equilibrium elasticities in order to gain a better understanding of the underlying adjustment processes. Table 2 reports elasticities of relative factor prices with respect to the relative prices of goods,

$$\epsilon_{wp} = d \ln w / d \ln p = \epsilon_w / \epsilon_p \tag{14}$$

and elasticities of output substitution

$$\epsilon_{xp} = d \ln x / d \ln p = \epsilon_x / \epsilon_p, \tag{15}$$

where $x = x_1/x_2$, $w = w_1/w_2$, and $p = p_1/p_2$. Arc elasticities at $p = 1$ are calculated across the smallest price increase in the simulations. Note that technologies other than those pictured are presented for contrast.

For instance, p is varied under t_1 from 1 to 1.211 in Table 1 to derive $\epsilon_{wp} = 1.78$ and $\epsilon_{xp} = 1.98$ in Table 2. Arc elasticities would vary for different levels and discrete changes of p , but inspection of the figures verifies the results of this section.

With CRS in both sectors, there is very high but sharply decreasing factor price elasticity as the difference across production functions increases. The RPL flattens as the CC becomes more convex. The factor price elasticities and the output elasticities are both high in t_1 .

PROPOSITION 9. With constant returns to scale in both sectors, factor price elasticities and output elasticities are both highly elastic for a wide range of relative prices.

With IRS in one sector, the negatively sloped RPL is inelastic. A higher p causes the relative output of good 1 to fall along the CC and the relative price of factor 1 to fall. In t_2 , the dual equilibrium gives rise to two sets of elasticities. In $t = (2.1 .5 .8 .2)$, the factor price elasticity is very small, but the output elasticity maintains a high magnitude.

With IRS in both sectors, factor price elasticities are negative but very inelastic when factor intensity differences are small. With a large factor intensity difference, the negative factor price elasticity becomes larger. Output elasticities are negative and remain large in magnitude.

PROPOSITION 10. Increasing returns to scale in both sectors implies negative and very elastic output elasticities, while factor price elasticities are negative but much smaller in magnitude for a range of relative prices. Dual stable equilibria may exist.

Table 2. Factor Price and Output Elasticities

					<i>wp</i>	<i>xp</i>			
CRS-CRS	t_1	.6	.4	.4	.6	5.00	23.77		
		.7	.3	.3	.7	2.99	5.21		
		.8	.2	.2	.8	1.78	1.98		
		.8	.2	.5	.5	3.33	8.97		
		.8	.2	.7	.3	10.00	73.25		
IRS-CRS	t_2	.9	.5	.5	.5	-0.59	-5.26		
		.9	.5	.2	.8	-2.28	4.81	-7.12	18.56
		2.1	.5	.8	.2	-0.02	-2.34		
IRS-IRS	t_4	.9	.5	.8	.5	-0.08	-3.75		
		.9	.5	.5	.8	-0.98	-4.70		
DRS-CRS	t_5	.5	.1	.5	.5	0.91	1.89		
		.5	.1	.2	.8	1.05	0.89		
		.5	.1	.8	.2	0.15	2.45		
DRS-DRS	t_6	.5	.1	.5	.2	0.43	1.69		
		.5	.1	.2	.5	0.97	0.84		
DRS-IRS	t_7	.5	.1	.9	.5	-0.40	0.46	-4.24	2.48
		.5	.1	.5	.9	-0.25	1.14	-3.41	1.98

With DRS in one sector, factor price elasticities are positive. The three sets of elasticities in this group reflect the curvature of the RPLs and PPFs. Factor price elasticities are typically inelastic, and output elasticities are relatively small. A larger factor price elasticity, created by larger differences in factor intensity across sectors, leads to smaller output elasticities with DRS in one sector.

With DRS in both sectors, the factor price elasticities in t_6 are positive, inelastic, and higher with a larger factor intensity difference. Again, a larger factor price elasticity means a smaller output elasticity. Compared with the CRS group, elasticities in the DRS group are much less elastic.

In the mixed DRS-IRS group, the sign of the factor price elasticities depends on the economy's position along its PPF. A relative price of 1 is consistent with two points on the PPF and two points on the RPL. The factor price elasticity is positive where the PPF is concave, and negative where it is convex. In the concave region of the PPF, increasing the difference in returns to scale across sectors causes the factor price elasticity to become larger and the output elasticity to become smaller. The

opposite is true (in absolute value) when the PPF is convex.

PROPOSITION 11. *With exponential production functions, outputs are generally sensitive to changes in the prices of goods. Output elasticities, in other words, are generally elastic. Factor prices are generally less sensitive to changes in the prices of goods. Decreasing returns to scale has a moderating effect on both factor price elasticities and output elasticities.*

8. CONCLUSION

This paper relates different exponential production technologies to the contract curve, production possibilities frontier, and relative price line in the general equilibrium model of production with two factors and two goods. Depending on the underlying technologies, exogenous changes in the relative prices of goods have different global effects on outputs and factor prices.

Underlying adjustment in factor prices to changes in the prices of goods varies significantly with technology. Elasticities of relative factor prices with respect to the relative prices of goods are elastic under constant returns to scale. Output elasticities are also typically elastic.

Decreasing returns to scale at the sectoral level have a moderating effect on the size of these elasticities. Casual empiricism suggests that factor prices and outputs are not outrageously sensitive to changes in the prices of goods. Decreasing returns to scale, however, are often dismissed in theoretical investigations, while increasing returns receive more attention.

This line of research can be directly extended to include many goods and factors of production. Models with three factors or three goods can be graphed. Different functional forms (constant elasticity of substitution, translog, and so on) can also be analyzed. In applied analysis, estimated production functions can be pictured.

This type of global analysis offers insight into the properties of fundamental general equilibrium models of production. It is worth emphasizing that contract curves as typically drawn with constant returns to scale are consistent with nearly linear production possibility frontiers. Production possibility frontiers as typically drawn imply radically convex contract curves.

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