Graph Theory Prelim, June 18, 2016

1) Prove that the following two statements are equivalent for the digraph D on n vertices:

a) D contains no consistently directed cycles.

b) The vertices of D may be labeled $v_1, v_2, \ldots, v_n$ in such a way that for all arcs e of D, if e is directed from v_i to v_j, then $i < j$.

2) a) Prove that if every vertex of a graph has degree > 1, it must contain a cycle.

b) Prove that if a graph has at least as many edges as vertices, it must contain a cycle.

3) Let G be a plane graph.

a) State Euler's formula relating the number of vertices v(G), the number of edges e(G), the number of faces f(G), and the number of components c(G).

b) Prove, possibly using a), that if each vertex of G has degree at least 3, then G must have a face bounded by at most 5 edges.

4) Let G be a simple graph, and $S \subseteq V(G)$. We say S is *independent* if no edge of G has both ends in S, and S is a *cover* if every edge of G has at least one end in S. We write *$\alpha(G)$* for the size of a largest independent set, and *$\beta(G)$* for the size of a smallest cover. A *minimum cover* is a cover of size $\beta(G)$.

a) Prove that $\alpha(G) + \beta(G) = |V(G)|$.

b) Prove that if x is a vertex or edge of G, then $\beta(G) - 1 \leq \beta(G \backslash \{x\}) \leq \beta(G)$.

c) We say that $U \subseteq V(G)$ is *special* in G if it is contained in no minimum cover of G, and *extra special* if it is special in G, but no proper subset of U is special in G. The graph G:U is obtained from G by adding a new vertex $y \notin V(G)$ adjacent in G:U precisely to the vertices in U.

i) Prove that $\beta(G{:}U) = \beta(G) + 1$ if and only if U is special in G.

ii) Prove that if U is extra special in G, then $\beta((G{:}U)\backslash\{e\}) = \beta(G)$ for every edge e of G:U incident with y.