Mathematics Department
Topology Prelim Exam
2:30 - 6:00 pm ± Aug. 31, 2024

Instructions: Do nine of the following problems at least four of which are asterisked. Use a separate answer sheet for each problem and put your name and problem number on each answer sheet. Fill in the "cover sheet" and place it at the beginning of your solutions.

Unless otherwise stated, assume that all spaces are Hausdorff.

Notation: If X is a space and $M \subset X$ then $\overline{M}$ denotes the closure of M in X.

Problem 1. Suppose that X is a set and $\mathcal{T}_X = \{H \subset X | X - H \text{ is finite }\} \cup \{\emptyset\}$.
(a) Argue that $\mathcal{T}_X$ is a topology for X,
(b) Argue that X is compact,
(c) Show that if X is infinite then X is connected.

Problem 2. Let X denote the real numbers. A basis for the Sorgenfrey topology $\mathcal{T}_S$ of X consists of all sets in the form $[a, b) = \{x | a \leq x < b\}$ for $a < b \in X$. Show:
(a.) If $\mathcal{T}_M$ is the (usual) metric topology for X then $\mathcal{T}_M \subset \mathcal{T}_S$.
(b.) In the Sorgenfrey topology X has no nondegenerate connected sets.

Problem 3. Suppose that X is a compact Hausdorff space. Prove that the common part of a monotonic subcollection G of nonempty closed sets is non-empty.

Problem 4. Use the results of problem 3 to prove that if each element of G is connected then so is the common part.

Problem 5. Prove that a space is regular if and only if for each closed set H and open set U containing H, there is an open set V so that

$$H \subset V \subset \overline{V} \subset U.$$

Problem 6*. Prove that a compact Hausdorff space in regular.

Problem 7*. Prove that a separable metric space has a countable basis.

Problem 8*. Prove that a metric space is
(a.) Normal,
(b.) First countable.

Problem 9. Suppose that M is a well ordered set with order $<$ and that M is uncountable but every initial segment is countable. Suppose that M is given the order topology and that Let L denotes the set of limit points of M. Show that M is order isomorphic to L.

Problem 10. Suppose that M is a well ordered set with order $<$ and that M has a last element. Show that M with the order topology is compact.

Problem 11*. Let X be a covering space for Y with covering map p and let $p(x_0) = y_0$; let $h : [0,1] \to Y$ be a path with initial point y_0. Prove that there is a unique lifting $\widetilde{h}$ of h to X so that $\widetilde{h}(0) = x_0$.

Problem 12. Prove that if X is homotopic to a point then the fundamental group of X is trivial.

Problem 13*. Prove that the Fundamental Group of the circle is the integers $(\mathbb{Z}, +)$.

Topology Prelim Exam Cover Page

Name: ____________________

ID Number: ____________________

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Circle the numbers of the problems you have selected to solve:

1 2 3 4 5 6* 7* 8* 9 10 11* 12 13*