Mathematical theories for metamaterials: From condensed matter theory to subwavelength physics

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Condensed matter physics

- Quantum condensed matter physics:
  - Concerned with situations where quantum physics and many-body interactions play a key role to create new physical phenomena.
  - Topological defects; Phase transitions; Hall effect; Localised states: Thouless, Duncan, Haldane, Kosterlitz, Anderson.
- Mathematical models:
  - Systems of particles;
  - Hamiltonians; Tight-binding model coupled with nearest-neighbour approximation;
  - Mathematical analysis: Fefferman-Weinstein, Ablowitz, Fröhlich, Mayboroda, Zworski, ...

Tight-binding approximation

- Schrödinger equation: $\Big(H_{\rm at} + V\Big)\Psi = E\Psi$; $H_{\rm at} = \sum_i H_i$; H_i: Hamiltonian of the single atom i, V: potential describing the interactions between the atoms; E: energy.
- Ψ: sum over atomic wave functions

$$\Psi(x) = \sum_i \sum_n a_i^{(n)} \phi_i^{(n)}(x);$$

  $\phi_i^{(n)}$: atomic wave function on the site i corresponding to the energy $e_i^{(n)}$ at the $n^{\rm th}$ atomic level.
- Assumptions: $\phi_i^{(n)} = \phi^{(n)}(x - z_i)$; z_i: position of the atom i, $H_j \phi_i^{(n)} = e_i^{(n)} \phi_i^{(n)} \delta_{ij}$; $\int \phi^{(n)}(x - z_i)\overline{\phi^{(m)}}(x - z_j)\, \mathrm{d}x = \delta_{ij}\delta_{nm}$.
- Schrödinger equation: matrix equation for the amplitudes $a_i^{(n)}$.

Tight-binding approximation

- Consider only one atomic level for each atom:

$$e_i a_i + \sum_j a_j \underbrace{\int V(x)\phi(x-z_i)\overline{\phi}(x-z_j)\,\mathrm{d}x}_{:=V_{ij}} = E a_i;$$

e_i: atomic energy level on the site i and V_{ij}: matrix element of the Hamiltonian between the atomic sites i and j.

- Tight-binding model coupled with a nearest-neighbour approximation: $V_{ii} = 0$ and $V_{ij} = 0$ for $|i-j| > 1$

$$\underbrace{\begin{pmatrix} e_1 & V_{12} & & \\ V_{21} & e_2 & V_{23} & \\ & \ddots & \ddots & \ddots \\ & & V_{N(N-1)} & e_N \end{pmatrix}}_{:=H_{\mathrm{tb}}} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = E \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}.$$

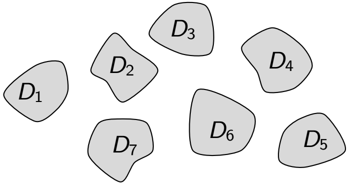
Subwavelength physics

- Subwavelength physics
  - Concerned with wave interactions with subwavelength structured materials.
  - Manipulate waves at subwavelength scales;
  - Subwavelength signal manipulation: revolutionising nanotechnology; applications in wireless communications, biomedical superresolution imaging and quantum computing.
- Physics and engineering literature: Tight-binding models.
- Mathematical Models:
  - Systems of subwavelength resonators; PDE models; Capacitance matrix approximations; Strong and long-range interactions in subwavelength resonator systems.
- Transpose demonstrated quantum phenomena to classical waves at subwavelength scales.
- First principle derivations from PDE models with long-range interactions.
- Mathematical theories for metamaterials: micro-structured materials with unusual properties.

Lecture I: Capacitance matrices and formulations

Capacitance matrix of a finite system

- Capacity of $D \subset \mathbb{R}^3$, bounded, connected domain with $C^{1,s}, 0 < s < 1$, boundary: $\mathrm{Cap}_D := \int_{\mathbb{R}^3 \setminus \overline{D}} |\nabla V|^2 \,\mathrm{d}x = -\int_{\partial D} \frac{\partial V}{\partial \nu}\Big|_+ \,\mathrm{d}\sigma$;
-
$$\begin{cases} \Delta V = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ V = 1 & \text{on } \partial D, \\ V(x) = \mathcal{O}\left(|x|^{-1}\right) & \text{as } |x| \to \infty. \end{cases}$$
- $D = D_1 \cup \cdots \cup D_N$; disjoint;
-
$$\begin{cases} \Delta V_i = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ V_i = \delta_{ij} & \text{on } \partial D_j, \\ V_i(x) = \mathcal{O}\left(|x|^{-1}\right) & \text{as } |x| \to \infty; \end{cases}$$



- Capacitance matrix of D: $C_{ij} := \int_{\mathbb{R}^3 \setminus \overline{D}} \nabla V_i \cdot \nabla V_j \,\mathrm{d}x = -\int_{\partial D_i} \frac{\partial V_j}{\partial \nu}\Big|_+ \,\mathrm{d}\sigma.$

Capacitance matrix of a finite system

- C: symmetric; positive definite;
- $C_{ij} < 0$ for any $1 \leq i \neq j \leq N$;
- C strictly diagonally dominant:

$$C_{ii} > \sum_{j \neq i} |C_{ij}|, \text{ for any } 1 \leq i \leq N;$$

- C: nonsingular Minkowski-matrix $\Rightarrow C^{-1}$: Minkowski-matrix; principle minors of C: positive.
- Dilute expansion: $D_i = \epsilon B_i + z_i, \epsilon \to 0$:

$$C_{ii} = \epsilon \mathrm{Cap}_{B_i} + \mathcal{O}(\epsilon^3),$$

$$C_{ij} = -\frac{\epsilon^2 \mathrm{Cap}_{B_i} \mathrm{Cap}_{B_j}}{4\pi |z_i - z_j|} + \mathcal{O}(\epsilon^3), \quad \text{for } i \neq j;$$

- Decay property for N large enough:

$$|C_{ij}| \lesssim \frac{1}{\mathrm{dist}(D_i, D_j)}.$$

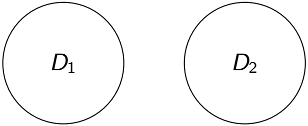
- $\Leftarrow C_{ij}^{(N)} \leq C_{ij}^{(N+1)}$; For $i = j \Rightarrow$ diagonal capacitance coefficients increase when adding additional resonators.

Capacitance matrix of a finite system

- Parity-symmetric system: Each resonator D_i can be uniquely associated to another resonator D_j (possibly with $i = j$) s.t. $\mathcal{P}D_i = D_j$; $\mathcal{P}(x) = -x$.
- $\Rightarrow C_{ii} = C_{jj}$.
- $N = 2$, $C_{11} = C_{22}, C_{12} = C_{21}$;

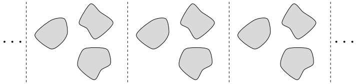
$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{11} \end{pmatrix};$$

- $\Rightarrow$ eigenvalues of C: $C_{11} + C_{12}$ and $C_{11} - C_{12}$; associated eigenvectors: $(1, 1)^\top, (-1, 1)^\top$.



Capacitance matrix of an infinite, periodic system

- d_l: dimension of periodicity of the lattice. d: dimension of the ambient space.
- Three different cases:
  - $d - d_l = 0$: crystal;
  - $d - d_l = 1$: screen;
  - $d - d_l = 2$: chain.

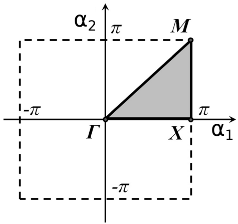


- Λ: periodic lattice; Y: fundamental domain; Λ^*: dual lattice of Λ; Brillouin zone $Y^* := \left(\mathbb{R}^{d_l} \times \{\mathbf{0}\}\right)/\Lambda^*$; $\mathbf{0}$: zero-vector in $\mathbb{R}^{d-d_l}$; $x = (x_l, x_0)$.
- Periodically repeated i^{th} $\mathcal{D}_i$ and the full periodic structure $\mathcal{D}$:

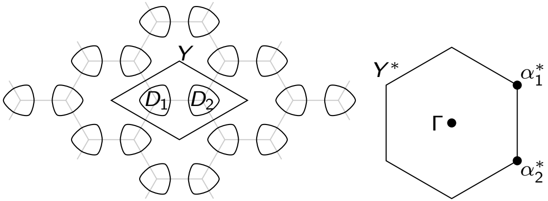
$$\mathcal{D}_i = \bigcup_{m \in \Lambda} D_i + m, \qquad \mathcal{D} = \bigcup_{i=1}^{N} \mathcal{D}_i.$$

Capacitance matrix of an infinite, periodic system

- Square lattice and corresponding Brillouin zone:



- Honeycomb lattice and corresponding Brillouin zone:



Capacitance matrix of an infinite, periodic system

- $f(x) \in L^2(\mathbb{R}^d)$: α-quasiperiodic, with quasiperiodicity $\alpha \in Y^*$, if $e^{-\mathrm{i}\alpha\cdot x} f(x)$: Λ-periodic;
- Floquet transform of $f \in L^2(\mathbb{R}^d)$:

$$\mathcal{U}[f](x,\alpha) := \sum_{m\in\Lambda} f(x-m)e^{\mathrm{i}\alpha\cdot m}, \quad x,\alpha \in \mathbb{R}^d.$$

- $\mathcal{U}[f]$: α-quasiperiodic in x and periodic in α.
- Floquet transform: invertible map $\mathcal{U} : L^2(\mathbb{R}^d) \to L^2(Y \times Y^*)$, with inverse given by

$$\mathcal{U}^{-1}[g](x) = \frac{1}{|Y_l^*|}\int_{Y^*} g(x,\alpha)\,\mathrm{d}\alpha, \quad x \in \mathbb{R}^d,$$

$g(x,\alpha)$: extended quasiperiodically for x outside of the unit cell Y.

Capacitance matrix of an infinite, periodic system

- Quasiperiodic capacitance matrix for $\alpha \in Y^*, \alpha \neq 0$:

$$C_{ij}^\alpha := \int_{Y\setminus D} \overline{\nabla V_i^\alpha} \cdot \nabla V_j^\alpha \,\mathrm{d}x, \quad i,j = 1,\ldots,N;$$

- $$\begin{cases} \Delta V_i^\alpha = 0 & \text{in } Y \setminus \overline{D}, \\ V_i^\alpha = \delta_{ij} & \text{on } \partial D_j, \\ V_i^\alpha(x+l) = e^{\mathrm{i}\alpha\cdot l} V_i^\alpha(x) & \forall l \in \Lambda, \\ V_i^\alpha(x) \to 0 & \text{as } |x_0| \to \infty, \end{cases}$$

  with $x = (x_l, x_0)$.

- C^α : Hermitian; positive definite.
- Dilute expansion: $D_i = \epsilon B_i + z_i, \epsilon \to 0$:

$$C_{ii}^\alpha = \epsilon \mathrm{Cap}_{B_i} - (\epsilon \mathrm{Cap}_{B_i})^2 \sum_{m\in\Lambda, m\neq 0} \frac{e^{\mathrm{i}m\cdot\alpha}}{4\pi|m|} + \mathcal{O}(\epsilon^3),$$

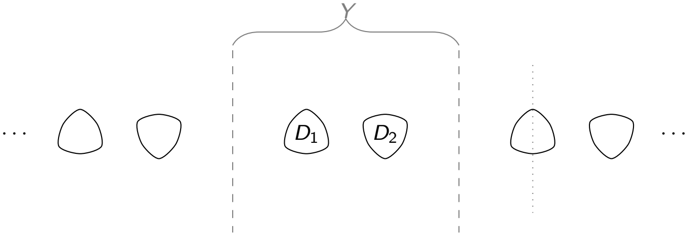
$$C_{ij}^\alpha = -\epsilon^2 \mathrm{Cap}_{B_i} \mathrm{Cap}_{B_j} \sum_{m\in\Lambda} \frac{e^{\mathrm{i}m\cdot\alpha}}{4\pi|m+z_i-z_j|} + \mathcal{O}(\epsilon^3), \quad \text{for } i \neq j.$$

Capacitance matrix of an infinite, periodic system

- Parity-symmetric system: In Y, each resonator D_i can be uniquely associated to another resonator D_j (possibly with $i = j$) s.t. $\mathcal{P}D_i = D_j$; $\mathcal{P}(x) = -x$.
- $\Rightarrow C^\alpha_{ii} = C^\alpha_{jj}$.
- $N = 2$, $C^\alpha_{11} = C^\alpha_{22}$, $C^\alpha_{12} = \overline{C^\alpha_{21}} \Rightarrow$ eigenvalues of C^α: $C^\alpha_{11} + |C^\alpha_{12}|$ and $C^\alpha_{11} - |C^\alpha_{12}|$; associated eigenvectors:

$$(e^{\mathrm{i}\theta_\alpha}, 1)^\top, (-e^{\mathrm{i}\theta_\alpha}, 1)^\top; e^{\mathrm{i}\theta_\alpha} = C^\alpha_{12}/|C^\alpha_{12}|;$$

α s.t. $C^\alpha_{12} \neq 0$.



“Real-space” capacitance matrix

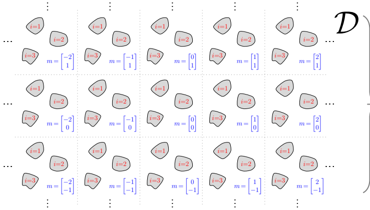
- C^α : “dual-space” representation of the infinite periodic system.
- Inverse Floquet transform ⇒ “real-space” capacitance matrix at $m \in \Lambda$:

$$\widehat{C}_{ij}^m = \frac{1}{|Y^*|} \int_{Y^*} C_{ij}^\alpha e^{-\mathrm{i}\alpha \cdot m} \,\mathrm{d}\alpha, \quad 1 \le i,j \le N.$$

- $\mathfrak{C}$: infinite matrix that contains all the $\widehat{C}_{ij}^m$ coefficients, for all $1 \le i,j \le N$ and all $m \in \Lambda$:

$$\mathfrak{C} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \widehat{C}^0 & \widehat{C}^1 & \widehat{C}^2 & \widehat{C}^3 & \cdots \\ \cdots & \widehat{C}^{-1} & \widehat{C}^0 & \widehat{C}^1 & \widehat{C}^2 & \cdots \\ \cdots & \widehat{C}^{-2} & \widehat{C}^{-1} & \widehat{C}^0 & \widehat{C}^1 & \cdots \\ \cdots & \widehat{C}^{-3} & \widehat{C}^{-2} & \widehat{C}^{-1} & \widehat{C}^0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Convergence of capacitance coefficients



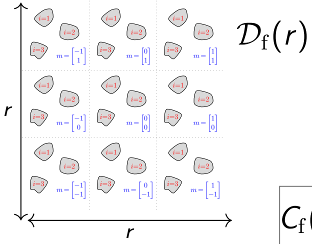
$C^\alpha, \alpha \in Y^*$

quasiperiodic capacitance matrix

Inverse Floquet transform

$\mathfrak{C}$

real-space capacitance matrix



$C_\mathrm{f}(r), r \in (0, \infty)$

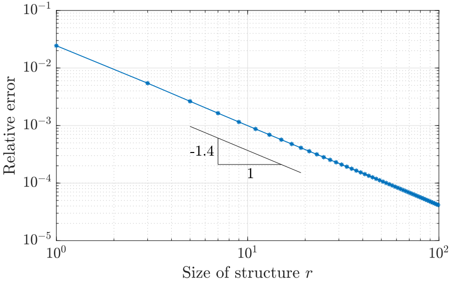
finite capacitance matrix

$C_\mathrm{t}(r), r \in (0, \infty)$

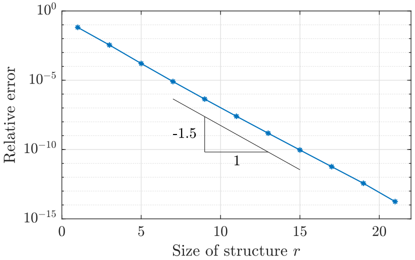
truncated capacitance matrix

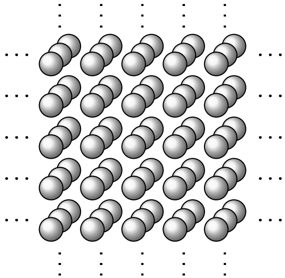
Convergence of capacitance coefficients

- Convergence of capacitance coefficients: For fixed $m, n \in \Lambda$, as $r \to \infty$,
$$\lim_{r\to\infty} C_{\mathrm{f}}^{mn}(r) = \widehat{C}^{m-n}.$$
- $|(C_{\mathrm{f}})_{11}^{0} - \widehat{C}_{11}^{0}|$ for increasing size r of the finite structure: algebraic ($d_l < d$)/exponential ($d = d_l$) convergence.
- $d_l < d$: long range interactions in the "spare" dimensions.



Convergence of capacitance coefficients





Convergence of capacitance coefficients

- $C_{\rm t}$: Toeplitz matrix of an essentially bounded symbol;
- As $r \to \infty$, the matrices $C_{\rm t}$ and $C_{\rm f}$ are asymptotically equivalent:
  - $\lim_{r\to\infty} |C_{\rm f} - C_{\rm t}| = 0$;
  - $\|C_{\rm f}\|_2$ and $\|C_{\rm t}\|_2$ are uniformly bounded as $r \to \infty$.
- For an $n \times n$ matrix $M = (m_{ij})$, normalised Frobenius norm:

$$|M|^2 = \frac{1}{n} \sum_{i,j=1}^{n} |m_{ij}|^2.$$

- Asymptotically equivalent matrices have identical eigenvalue distributions as their sizes tend to infinity.

Periodic capacitance matrix

- Fix $\alpha_0 : |\alpha_0| = 1$; $V_i^\alpha \to V_i^0$ as $\alpha = |\alpha|\alpha_0 \to 0$:

$$\begin{cases} \Delta V_i^0 = 0 & \text{in } Y \setminus D, \\ V_i^0 = \delta_{ij} & \text{on } \partial D_j, \\ V_i^0(x_l, x_0) & \text{is } \Lambda\text{-periodic in } x_l, \\ V_i^0(x_l, x_0) \to \pm V_\infty^i & \text{as } x_0 \to \pm\infty; \end{cases}$$

- V_∞^i constants; may depend on α_0.
- Periodic capacitance matrix:

$$C_{ij}^0 = \int_{Y \setminus D} \nabla V_i^0 \cdot \nabla V_j^0 \, \mathrm{d}x.$$

- C^0: real, symmetric, positive semi-definite matrix with one vanishing eigenvalue.
- C^0: independent of α_0 for parity symmetric dimer of resonators.

Asymptotic perturbation theory of Gohberg-Sigal

- $A(z)$ finitely meromorphic of Fredholm type at z_0:

$$A(z) = \sum_{j \geq -s} (z - z_0)^j A_j;$$

$A_{-j}, j = 1, \ldots, s$: finite-dimensional ranges and A_0: Fredholm.

- z_0: normal point of $A(z)$ if $A(z)$: finitely meromorphic, of Fredholm type at z_0, holomorphic, and invertible in a neighborhood of z_0 except at z_0 itself.
- V: simply connected bounded domain with rectifiable boundary ∂V; $A(z)$: normal with respect to ∂V if $A(z)$: finitely meromorphic and of Fredholm type in V, continuous on ∂V, and invertible for $z \in \overline{V}$, except for a finite number of points of V which are normal points of $A(z)$.

Asymptotic perturbation theory of Gohberg-Sigal

- z_0: characteristic value of A if there exists a vector-valued function $\phi(z)$ with values in $\mathcal{B}$ s.t.
  (i) $\phi(z)$: holomorphic at z_0 and $\phi(z_0) \neq 0$;
  (ii) $A(z)\phi(z)$: holomorphic at z_0 and vanishes at this point.
- $\phi(z)$: root function of $A(z)$ associated with the characteristic value z_0.
- There exists $m(\phi) \geq 1$ and a vector-valued function ψ holomorphic at z_0 s.t.

$$A(z)\phi(z) = (z - z_0)^{m(\phi)}\psi(z), \quad \psi(z_0) \neq 0.$$

- $m(\phi)$: multiplicity of the root function $\phi(z)$.
- $\phi_0 \in \mathrm{Ker}\, A(z_0)$; $\mathrm{rank}(\phi_0)$: maximum of the multiplicities of all root functions $\phi(z)$ with $\phi(z_0) = \phi_0$; ϕ_0 : eigenvector.

Asymptotic perturbation theory of Gohberg-Sigal

- Assumptions: $n = \dim \operatorname{Ker} A(z_0) < +\infty$; ranks of all vectors in $\operatorname{Ker} A(z_0)$: finite.
- Canonical system of eigenvectors of A associated to z_0: system of eigenvectors ϕ_0^j, $j = 1, \ldots, n$, s.t. for $j = 1, \ldots, n$, $\mathrm{r}ank(\phi_0^j)$: the maximum of the ranks of all eigenvectors in the direct complement in $\operatorname{Ker} A(z_0)$ of the linear span of the vectors $\phi_0^1, \ldots, \phi_0^{j-1}$.
- Null multiplicity of the characteristic value z_0 of A:

$$N(A(z_0)) := \sum_{j=1}^{n} \mathrm{r}ank(\phi_0^j).$$

- If z_0: not a characteristic value of A, we put $N(A(z_0)) = 0$.
- Multiplicity of z_0:

$$M(A(z_0)) = N(A(z_0)) - N(A(z_0)^{-1})$$

- z_0: characteristic value and not a pole of $A(z) \Rightarrow M(A(z_0)) = N(A(z_0))$; $M(A(z_0)) = -N(A(z_0)^{-1})$ if z_0: pole and not a characteristic value of $A(z)$.

Asymptotic perturbation theory of Gohberg-Sigal

- $A(z)$: normal with respect to ∂V and z_i, $i = 1, \ldots, \sigma$, are all its characteristic values and poles in V;
- Full multiplicity $\mathcal{M}(A; \partial V)$ of $A(z)$ for $z \in V$: number of characteristic values of $A(z)$ for $z \in V$, counted with their multiplicities, minus the number of poles of $A(z)$ in V, counted with their multiplicities.
- Generalised argument principle: $A(z)$: normal with respect to ∂V; $f(z)$: holomorphic in V and continuous in $\overline{V}$;

$$\frac{1}{2\pi \mathrm{i}} \, \mathrm{tr} \int_{\partial V} f(z) A(z)^{-1} \frac{d}{dz} A(z) dz = \sum_{j=1}^{\sigma} M(A(z_j)) f(z_j).$$

- Generalised Rouché's theorem: $A(z)$: normal with respect to ∂V; $S(z)$: finitely meromorphic in V and continuous on ∂V s.t.

$$\|A(z)^{-1} S(z)\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} < 1, \qquad z \in \partial V.$$

$\Rightarrow A + S$: normal with respect to ∂V and

$$\mathcal{M}(A; \partial V) = \mathcal{M}(A + S; \partial V).$$

Abstract capacitance matrix

- $\mathcal{H}, \mathcal{H}'$: Hilbert spaces; $\mathcal{L}(\mathcal{H}, \mathcal{H})$: space of bounded linear operators from $\mathcal{H}$ into $\mathcal{H}'$;
- Operator-valued function $\omega \mapsto \mathcal{A}(\omega, \delta) \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$;
- $\mathcal{A}(\omega, \delta)$: Fredholm of index zero, holomorphic with respect to ω and δ.
- $\mathcal{A}(\omega, 0)$ has a characteristic value $\omega = 0$ of multiplicity $2N$, admitting the pole-pencil decomposition:

$$\mathcal{A}(\omega, 0)^{-1} = \frac{K}{\omega^2} + \mathcal{R}(\omega), \quad \text{for} \quad K = \sum_{i=1}^{N} \langle \Phi_i, \cdot \rangle \Psi_i;$$

- $\mathrm{Ker}\, \mathcal{A}(0,0) = \mathrm{span}\{\Psi_j\}$, $\mathrm{Ker}\, \mathcal{A}^*(0,0) = \mathrm{span}\{\Phi_j\}$; $\mathcal{R}$: holomorphic for ω in a neighbourhood of 0; $\mathcal{A}^*$: adjoint of $\mathcal{A}$.
- $\mathcal{A}(\omega, \delta)$, for small but nonzero δ, satisfies

$$\mathcal{A}(\omega, \delta) = \mathcal{A}(\omega, 0) + \mathcal{L}(\omega, \delta),$$

  for some operator $\mathcal{L}$ satisfying (in corresponding operator norm) $\|\mathcal{L}\| = \mathcal{O}(\delta)$ uniformly for ω in a neighbourhood of 0.

Abstract capacitance matrix

- Generalised Rouché's theorem $\Rightarrow \mathcal{A}(\omega,\delta)$: $2N$ characteristic values in a small neighborhood of 0.
- Asymptotic formulas (in δ) of the characteristic values: $\mathcal{A}(\omega,\delta)[\Phi]=0$.
- Multiplying with $\mathcal{A}(\omega,0)^{-1}$, we have

$$0=\mathcal{A}(\omega,0)^{-1}\mathcal{A}(\omega,\delta)[\Phi]=\mathcal{A}(\omega,0)^{-1}\left(\mathcal{A}(\omega,0)+\mathcal{L}\right)[\Phi]=\left(I+\frac{K\mathcal{L}}{\omega^2}+\mathcal{R}\mathcal{L}\right)\Phi.$$

- Defining $\mathcal{B}(\omega,\delta)=\omega^2\mathcal{R}(\omega)\mathcal{L}(\omega,\delta)\Rightarrow$

$$\left(\omega^2 I+K\mathcal{L}+\mathcal{B}\right)[\Phi]=0.$$

- Characteristic values of $\mathcal{A}(\omega,\delta)$: determined by a nonlinear eigenvalue problem since $\mathcal{L}$ and $\mathcal{B}$ depend on ω.
- For small ω, we have $\|\mathcal{B}\|=\mathcal{O}(\omega^2\delta)$ uniformly for ω and δ around 0;
- $\mathcal{L}=\mathcal{L}_0+\widehat{\mathcal{L}}$, $\mathcal{L}_0$: independent of ω and $\widehat{\mathcal{L}}=\mathcal{O}(\omega\delta)$.

Abstract capacitance matrix

- Characteristic values pf $\mathcal{A}(\omega, \delta)$ in a small neighborhood of 0 approximated by $\pm$ the square roots of the eigenvalues of the finite-rank operator $-K\mathcal{L}_0$;
- Restriction of $-K\mathcal{L}_0$ to $\mathrm{Ker}\, \mathcal{A}(0,0)$ given by the N by N matrix:

$$\mathcal{C}_{ij} = -\langle \Phi_i, \mathcal{L}_0[\Psi_j] \rangle.$$

- Characteristic values of $\mathcal{A}(\omega, \delta)$:

$$\omega_n = \pm\sqrt{\lambda_n} + \mathcal{O}(\delta);$$

- λ_n: eigenvalues of $\mathcal{C}$.
- $\omega_n = \mathcal{O}(\sqrt{\delta})$ since $\|\mathcal{L}_0\| = \mathcal{O}(\delta)$.

Abstract capacitance matrix

- Pole-pencil decomposition:

$$\mathcal{A}(\omega, 0)^{-1} = \frac{K + \omega \mathcal{R}^{(1)}}{\omega^2} + \mathcal{R}^{(2)}(\omega),$$

- $\mathcal{R}^{(1)}$: independent of ω; $\mathcal{R}^{(2)}$: holomorphic in ω in a neighborhood of 0.
- Assume

$$\mathcal{A}(\omega, \delta) = \mathcal{A}(\omega, 0) + \mathcal{L}(\omega, \delta), \quad \mathcal{L} = \mathcal{L}_0 + \widehat{\mathcal{L}};$$

- $\|\mathcal{L}\| = \mathcal{O}(\delta), \|\widehat{\mathcal{L}}\| = \mathcal{O}(\omega^2 \delta)$ uniformly for ω in a neighbourhood of 0.
- $\Rightarrow$

$$\left(\omega^2 I + K\mathcal{L}_0 + \omega \mathcal{R}^{(1)} \mathcal{L}_0 + \mathcal{B}\right)[\Phi] = 0;$$

- $\|\mathcal{B}\| = \mathcal{O}(\omega^2 \delta)$ uniformly for ω and δ around 0 $\Rightarrow$ under the assumption that all the eigenvalues λ_n of $\mathcal{C}$ are simple, the characteristic values of $\mathcal{A}(\omega, \delta)$:

$$\omega_n = \pm\sqrt{\lambda_n} + \langle \mathcal{R}^{(1)} \mathcal{L}_0[v_n], v_n \rangle + \mathcal{O}(\delta^{3/2});$$

- v_n: normalised eigenvectors associated to the eigenvalues λ_n.
- Correction term of order δ since $\|\mathcal{L}_0\| = \mathcal{O}(\delta)$.

Open questions

- Shape derivative of the capacitance matrix C;
- Isoperimetric inequalities for $\mathrm{tr}(C)$; Capacitance matrix of the Minkwoski sum: $C(tD_1 + (1-t)D_2), 0 \leq t \leq 1.$
- Equivalent representation of a system of arbitrary shaped resonators by spherical resonators;
- Algebraic/exponential rate of convergence of the capacitance coefficients as the size of the corresponding system goes to ∞.

Lecture II: Subwavelength resonances

Subwavelength resonances

- Functional analytic approach to characterise a finite system of subwavelength resonators;
- Discrete approximation to subwavelength scattering and resonance problems in terms of the generalised capacitance matrix;
- Leading-order asymptotic expressions for both resonant modes and scattered solutions in terms of its eigenvalues and eigenvectors, which are accompanied by precise error bounds.
- Integral approach with rigorous justification based on the asymptotic perturbation theory of Gohberg and Sigal;
- Relate the capacitance matrix formalism to the tight-binding approximation in condensed matter physics.

M. Fink et al.

Scattering problem

- $D_1, D_2, \ldots, D_N \subset \mathbb{R}^d$, $d \in \{2,3\}$, $N \in \mathbb{N}$: disjoint, connected sets with boundaries in $C^{1,s}$ for some $0 < s < 1$.
- v_i: wave speed in resonator D_i; $k_i = \omega / v_i$: wave number in D_i, where $\omega \in \mathbb{R}, \omega \neq 0$,: operating frequency; v and k: wave speed and wave number in the background medium.
- Scattering problem:

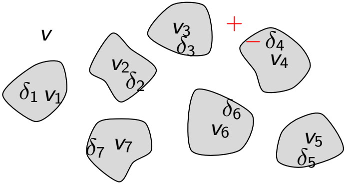
$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^d \setminus \overline{D}, \\ \Delta u + k_i^2 u = 0 & \text{in } D_i, \text{ for } i = 1, \ldots, N, \\ u|_+ - u|_- = 0 & \text{on } \partial D, \\ \delta_i \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial D_i \text{ for } i = 1, \ldots, N, \\ u - u_{\mathrm{in}} \text{ satisfies an outgoing radiation condition.} \end{cases}$$

- High contrast regime $0 < \delta \ll 1$:

$$v, v_i = \mathcal{O}(1), \delta_i = \mathcal{O}(\delta), \qquad \text{for } i = 1, \ldots, N.$$

Subwavelength resonance problem

- Finite collection of resonators:



- Subwavelength resonant frequency: Given $\delta > 0$, a subwavelength resonant frequency $\omega = \omega(\delta) \in \mathbb{C}$:
  - (i) there exists a non-trivial solution to the scattering problem with $u_{\text{in}} \equiv 0$, known as an associated resonant mode;
  - (ii) ω depends continuously on δ and satisfies $\omega \to 0$ as $\delta \to 0$.

Boundary integral formulation

- Helmholtz Green's function:

$$G^\omega(x) = \begin{cases} -\dfrac{\mathrm{i}}{4} H_0^{(1)}(\omega|x|), & d=2, \\ -\dfrac{1}{4\pi|x|} e^{\mathrm{i}\omega|x|}, & d=3, \end{cases} \qquad x \neq 0,\ \Re(\omega) > 0.$$

- Single layer potential $\mathcal{S}_D^\omega : L^2(\partial D) \to H^1_{\mathrm{loc}}(\mathbb{R}^d)$:

$$\mathcal{S}_D^\omega[\varphi](x) = \int_{\partial D} G^\omega(x-y)\varphi(y)\,\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^d,\ \varphi \in L^2(\partial D).$$

- Neumann–Poincaré operator $\mathcal{K}_D^{\omega,*} : L^2(\partial D) \to L^2(\partial D)$:

$$\mathcal{K}_D^{\omega,*}[\varphi](x) = \int_{\partial D} \frac{\partial}{\partial \nu_x} G^\omega(x-y)\varphi(y)\,\mathrm{d}\sigma(y), \quad x \in \partial D,\ \varphi \in L^2(\partial D).$$

- Jump relations:

$$\mathcal{S}_D^\omega[\varphi]\big|_+ = \mathcal{S}_D^\omega[\varphi]\big|_-, \quad \frac{\partial}{\partial \nu}\mathcal{S}_D^\omega[\varphi]\Big|_\pm = \left(\pm\frac{1}{2}I + \mathcal{K}_D^{\omega,*}\right)[\varphi].$$

Boundary integral formulation

- Subwavelength resonance problem is equivalent to finding $\omega(\delta)$ s.t. $\omega(\delta) \to 0$ as $\delta \to 0$ and there exists a non-trivial pair of densities $(\phi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ s.t.

$$\mathcal{A}(\omega, \delta) \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

- $\mathcal{A}(\omega, \delta) : L^2(\partial D) \times L^2(\partial D) \to H^1(\partial D) \times L^2(\partial D)$:

$$\mathcal{A}(\omega, \delta) = \begin{pmatrix} \widetilde{\mathcal{S}}_D^{\omega} & -\mathcal{S}_D^{k} \\ -\frac{1}{2} I + \widetilde{\mathcal{K}}_D^{\omega,*} & -\widetilde{\delta}\left(\frac{1}{2} I + \mathcal{K}_D^{k,*}\right) \end{pmatrix}.$$

- Scattering problem: $\omega \neq 0$ s.t. k_i^2 is not Dirichlet eigenvalue for $-\Delta$ on D_i, $i = 1, \ldots, N$;

$$u(x) = \begin{cases} u_{\rm in}(x) + \mathcal{S}_D^{k}[\phi](x), & x \in \mathbb{R}^3 \setminus \overline{D}, \\ \widetilde{\mathcal{S}}_D^{\omega}[\psi](x), & x \in D, \end{cases}$$

- $(\phi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ unique solution of

$$\mathcal{A}(\omega, \delta) \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} u_{\rm in} \\ \tilde{\delta} \frac{\partial u_{\rm in}}{\partial \nu} \end{pmatrix} \quad \text{on } \partial D.$$

- $\widetilde{\delta}(x) = \delta_i, x \in \partial D_i$;

$$\widetilde{\mathcal{S}}_D^{\omega}[\varphi](x) = \mathcal{S}_D^{k_i}[\varphi](x); \quad \widetilde{\mathcal{K}}_D^{\omega,*}[\varphi](x) = \mathcal{K}_D^{k_i,*}[\varphi](x), \quad x \in \partial D_i, \ \varphi \in L^2(\partial D).$$

Capacitance formulation of the resonance problem

- Let $d = 3$; $\mathcal{H} = L^2(\partial D) \times L^2(\partial D), \mathcal{H}' = H^1(\partial D) \times L^2(\partial D)$; $\mathcal{A}(0,0) \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$:

$$\mathcal{A}(0,0) = \begin{pmatrix} \mathcal{S}_D^0 & -\mathcal{S}_D^0 \\ -\frac{1}{2}I + \mathcal{K}_D^{0,*} & 0 \end{pmatrix},$$

- Perturbations of $\mathrm{Ker}\, \mathcal{A}(0,0)$ when δ and ω are nonzero.
- $\mathcal{S}_D^0 : L^2(\partial D) \rightarrow H^1(\partial D)$ is invertible;
- $\mathrm{Ker}\,(-\frac{1}{2}I + \mathcal{K}_D^{0,*}) = \mathrm{span}\{\psi_1, \psi_2, \ldots, \psi_N\}$,

$$\psi_i := (\mathcal{S}_D^0)^{-1}[\chi_{\partial D_i}];$$

$\chi_{\partial D_i}$: characteristic function of ∂D_i, for $i = 1, \ldots, N$.

- $\Rightarrow \mathcal{A}(0,0)$: N-dimensional kernel $\Rightarrow$ $\omega = 0$: characteristic value of $\mathcal{A}(\omega, 0)$ of multiplicity $2N$.

Capacitance formulation of the resonance problem

- $\delta_i, v_i \in \mathbb{R}$ for all $i = 1, \ldots, N$; Symmetry of the set of subwavelength resonant frequencies with respect to the imaginary axis:

$$\mathcal{A}(-\overline{\omega}, \delta) \begin{pmatrix} \overline{\psi} \\ \overline{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- $\mathcal{A} \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$:

$$\mathcal{A}(\omega, 0) = \begin{pmatrix} \widetilde{\mathcal{S}}_D^{\omega} & -\mathcal{S}_D^k \\ -\frac{1}{2}I + \widetilde{\mathcal{K}}_D^{\omega,*} & 0 \end{pmatrix}, \quad \mathcal{L}(\omega, \delta) = \begin{pmatrix} 0 & 0 \\ 0 & -\widetilde{\delta}\left(\frac{1}{2}I + \mathcal{K}_D^{k,*}\right) \end{pmatrix};$$

$$\mathcal{L}_0(\delta) = \begin{pmatrix} 0 & 0 \\ 0 & -\widetilde{\delta}\left(\frac{1}{2}I + \mathcal{K}_D^{0,*}\right) \end{pmatrix}.$$

- $\|\mathcal{L}_0\| = \mathcal{O}(\delta)$; $\|\mathcal{L} - \mathcal{L}_0\| = \mathcal{O}(\omega^2 \delta)$.

Capacitance formulation of the resonance problem

- $\mathcal{A}(\omega, 0)^{-1}$ satisfies

$$\mathcal{A}(\omega, 0)^{-1} = \frac{K + \omega \mathcal{R}^{(1)}}{\omega^2} + \mathcal{R}^{(2)}(\omega);$$

- $$\Phi_i = -\frac{v_i^2}{|D_i|}\begin{pmatrix} 0 \\ \chi_{\partial D_i} \end{pmatrix}, \qquad \Psi_j = \begin{pmatrix} 0 \\ \psi_j \end{pmatrix}, \quad \psi_j = (\mathcal{S}_D^0)^{-1}[\chi_{\partial D_j}];$$
$$(\mathcal{R}^{(1)})_{11} = \frac{v_i}{|D_i|}(\mathcal{S}_D^0)^{-1}\mathcal{S}_{D,1}(\mathcal{S}_D^0)^{-1}\left[\chi_{\partial D_i}\right]\langle \chi_{\partial D_i}, \cdot \rangle;$$
$$(\mathcal{R}^{(1)})_{12} = \frac{v}{|D_i|}(\mathcal{S}_D^0)^{-1}\mathcal{S}_{D,1}(\mathcal{S}_D^0)^{-1}\left[\chi_{\partial D_i}\right]\langle \chi_{\partial D_i}, \cdot \rangle;$$
$$(\mathcal{R}^{(1)})_{21} = (\mathcal{R}^{(1)})_{22} = 0.$$

Capacitance formulation of the resonance problem

- Generalised Rouché's theorem $\Rightarrow$ for sufficiently small $\delta > 0$, there exist N subwavelength resonant frequencies $\omega_1(\delta), \ldots, \omega_N(\delta)$ with non-negative real parts.
- Generalised capacitance matrix:

$$\mathcal{C}_{ij} = -\frac{\delta_i v_i^2}{|D_i|} \langle \chi_{\partial D_i}, \psi_j \rangle = \frac{\delta_i v_i^2}{|D_i|} C_{ij}.$$

- C: capacitance matrix

$$C_{ij} = -\int_{\partial D_i} (\mathcal{S}_D^0)^{-1}[\chi_{\partial D_j}] \, \mathrm{d}\sigma, \quad i, j = 1, \ldots, N.$$

-

$$\mathcal{C} = VC, \qquad V = \begin{pmatrix} \frac{v_1^2 \delta_1}{|D_1|} & & \\ & \ddots & \\ & & \frac{v_N^2 \delta_N}{|D_N|} \end{pmatrix}$$

Capacitance formulation of the resonance problem

- As $\delta \to 0$, the N subwavelength resonant frequencies (with non-negative real parts) satisfy the asymptotic formula

$$\omega_n = \sqrt{\lambda_n} + \mathcal{O}(\delta), \quad n = 1, \ldots, N,$$

$\{\lambda_n : n = 1, \ldots, N\}$: eigenvalues of the generalised capacitance matrix $\mathcal{C} \in \mathbb{C}^{N \times N}$, which satisfy $\lambda_n = \mathcal{O}(\delta)$ as $\delta \to 0$.

- v_n: normalised eigenvector of $\mathcal{C}$ associated to the eigenvalue λ_n. Then the normalised resonant mode u_n associated to the resonant frequency ω_n is given, as $\delta \to 0$, by

$$u_n(x) = \begin{cases} v_n \cdot S_D^{\omega_n/v}(x) + \mathcal{O}(\delta^{1/2}), & x \in \mathbb{R}^3 \setminus \overline{D}, \\ v_n \cdot S_D^{\omega_n/v_i}(x) + \mathcal{O}(\delta^{1/2}), & x \in D_i, \end{cases}$$

$S_D^k : \mathbb{R}^3 \to \mathbb{C}^N$ vector-valued function given by

$$S_D^k(x) = \begin{pmatrix} \mathcal{S}_D^k[\psi_1](x) \\ \vdots \\ \mathcal{S}_D^k[\psi_N](x) \end{pmatrix}, \quad x \in \mathbb{R}^3 \setminus \partial D,$$

with $\psi_i := (\mathcal{S}_D^0)^{-1}[\chi_{\partial D_i}]$ for $i = 1, \ldots, N$.

Capacitance formulation of the resonance problem

- Suppose $v_1 = v_2 = \cdots = v_N$ and $\delta_1 = \delta_2 = \cdots = \delta_N$. As $\delta \to 0$, the N subwavelength resonant frequencies satisfy the asymptotic formula

$$\omega_n = \sqrt{\lambda_n} - \mathrm{i}\tau_n + \mathcal{O}(\delta^{3/2}), \quad n = 1, \ldots, N;$$

- λ_n for $n = 1, \ldots, N$: eigenvalues of the generalised capacitance matrix $\mathcal{C}$;
- τ_n:

$$\tau_n = \delta_1 \frac{v_1^2}{8\pi v} \frac{\mathbf{v}_n^\top CJC\mathbf{v}_n}{\|\mathbf{v}_n\|_D^2};$$

- C: capacitance matrix, J the $N \times N$ matrix of ones, $\mathbf{v}_n$ the eigenvector associated to λ_n; $\|x\|_D := \left(\sum_{i=1}^N |D_i| x_i^2\right)^{1/2}$ for $x \in \mathbb{R}^N$.
- For each $n = 1, \ldots, N$, it holds that $\sqrt{\lambda_n} = \mathcal{O}(\delta^{1/2})$ and $\tau_n = \mathcal{O}(\delta)$ as $\delta \to 0$.

Capacitance formulation of the resonance problem

- Single resonator:

$$\omega_1 = \underbrace{\sqrt{\frac{\mathrm{Cap}_D}{|D|}} v_r \sqrt{\delta}}_{:=\omega_M} - \mathrm{i} \underbrace{\left(\frac{\mathrm{Cap}_D^2 v_r^2}{8\pi v |D|} \delta \right)}_{:=\tau_M} + \mathcal{O}(\delta^{\frac{3}{2}});$$

- Monopole approximation:

$$u^s(x) := (u - u_{\mathrm{in}})(x) = g(\omega, \delta, D)(1 + o(1)) u_{\mathrm{in}}(0) G^k(x); \quad 0 \in D;$$

- Scattering coefficient:

$$g(\omega, \delta, D) = \frac{\mathrm{Cap}_D}{1 - (\frac{\omega_M}{\omega})^2 + \mathrm{i}\gamma_M};$$

- Damping constant:

$$\gamma_M := \frac{\omega(v + v_r)\mathrm{Cap}_D}{8\pi v v_r} - \frac{(v - v_r)}{v} \frac{\delta v_r \mathrm{Cap}_D^2}{8\pi |D| \omega}.$$

- Scattering enhancement near ω_M.

Capacitance formulation of the resonance problem

- Parity-symmetric dimer (with respect to 0):

$$\omega_1 = \underbrace{\sqrt{(C_{11} + C_{12})} v_r \sqrt{\delta}}_{:=\omega_{M,1}} - \mathrm{i}\tau_1 \delta + \mathcal{O}(\delta^{3/2});$$

$$\omega_2 = \underbrace{\sqrt{(C_{11} - C_{12})} v_r \sqrt{\delta}}_{:=\omega_{M,2}} + \delta^{3/2} \widehat{\eta}_1 + \mathrm{i}\delta^2 \widehat{\eta}_2 + \mathcal{O}(\delta^{5/2});$$

- $\widehat{\eta}_1, \widehat{\eta}_2$: real numbers determined by D, v, and v_r;

$$\tau_1 = \frac{v_r^2}{4\pi v} (C_{11} + C_{12})^2 .$$

- ω_1 and ω_2: monopole and dipole hybridised resonances of the resonator dimer D.
- $\omega_{M,1}$: slightly smaller than $\omega_{M,2}$; $\Im \omega_1 = \mathcal{O}(\delta)$ while $\Im \omega_2 = \mathcal{O}(\delta^2)$.

Capacitance formulation of the resonance problem

- Point scatterer with resonant monopole and resonant dipole modes:

$$u^s(x) = \underbrace{g^0(\omega) u_{\rm in}(0) G^k(x)}_{\text{monopole}} + \underbrace{\nabla u_{\rm in}(0) \cdot g^1(\omega) \nabla G^k(x)}_{\text{dipole}} + \mathcal{O}(\delta |x|^{-1}),$$

- $g^0(\omega), g^1(\omega) = (g^1_{ij}(\omega))$:

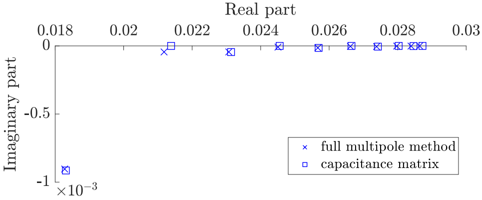
$$g^0(\omega) = \frac{C(1,1)}{1 - \omega_1^2/\omega^2}(1 + \mathcal{O}(\delta^{1/2})), \quad C(1,1) := C_{11} + C_{12} + C_{21} + C_{22},$$

$$g^1_{ij}(\omega) = \int_{\partial D} (\mathcal{S}_D^0)^{-1}[x_i](y) y_j - \frac{\delta v_r^2}{\omega^2 |D| (1 - \omega_2^2/\omega^2)} P^2 \delta_{i1}\delta_{j1},$$

$$P := \int_{\partial D} y_1 (\mathcal{S}_D^0)^{-1}(\chi_{\partial D_1} - \chi_{\partial D_2})(y) \, \mathrm{d}\sigma(y).$$

Capacitance formulation of the resonance problem

- Multipole expansion method + Muller's method: numerical (complex) root finding method using quadratic interpolants.
- Generalised capacitance matrix approximations: significant reduction in computational power.
- Subwavelength resonant frequencies of a system of ten spherical resonators:



Capacitance formulation of the resonance problem

- Let $d = 2$. A system of N subwavelength resonators in $\mathbb{R}^2$ has N subwavelength resonant frequencies.
- $\mathcal{A}^{(2)}_{\omega,\delta}$:

$$(\mathcal{A}^{(2)}_{\omega,\delta})_{ij} = \omega^2 \ln \omega + \left(\left(1 + \frac{c_1}{b_1} - \ln v_i \right) - \frac{\mathcal{S}_D^0[\psi_j]|_{\partial D_i}}{4 b_1 (\int_{\partial D} \psi_j)} \right) \omega^2 - \frac{v_i^2 \delta_i}{4 b_1 |D_i|} \left(\frac{\int_{\partial D_i} \psi_j}{(\int_{\partial D} \psi_j)} + \frac{\ln(v/v_i)}{2\pi} \int_{\partial D_i} (\widehat{\mathcal{S}}_D^k)^{-1}[\chi_{\partial D}] \right),$$

$b_1 = -\frac{1}{8\pi}, c_1 = b_1(\gamma - \ln 2 - 1 - \mathrm{i}\,\frac{\pi}{2})$;

- Subwavelength resonant frequencies: $\det \mathcal{A}^{(2)}_{\omega,\delta} = 0$ (at leading order):

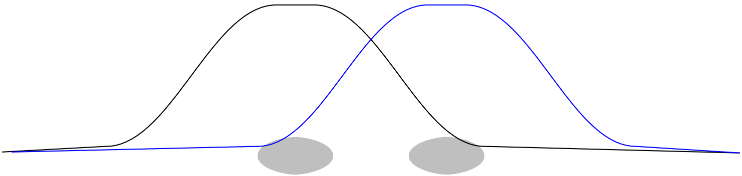
$$\det(\mathcal{A}^{(2)}_{\omega,\delta}) = \mathcal{O}(\omega^4 \ln \omega + \delta \omega^2 \ln \omega), \quad \text{as } \omega, \delta \to 0.$$

- $\mathcal{S}_D^0[\psi_j]$: constant in D_j since $\psi_j \in \mathrm{Ker}\,(-\frac{1}{2}I + \mathcal{K}_D^{0,*}) \Rightarrow -\dfrac{\mathcal{S}_D^0[\psi_j]|_{\partial D_j}}{\int_{\partial D} \psi_j} =$ $1/(2\pi)\times$ logarithm of the capacity of D_j.
- Two-dimensional analogue of the capacitance matrix:

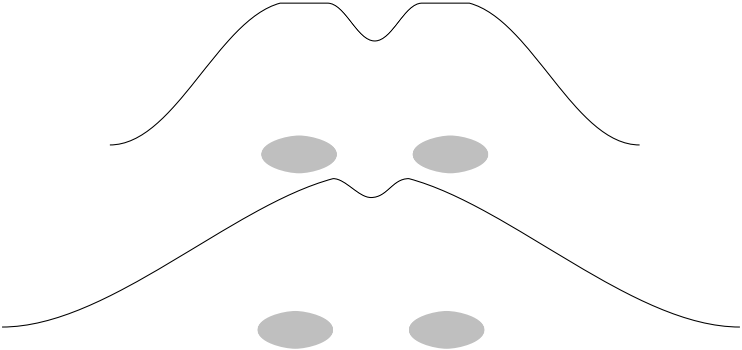
$$-\frac{\mathcal{S}_D^0[\psi_j]|_{\partial D_i}}{\int_{\partial D} \psi_j}.$$

Comparison between the tight-binding and capacitance matrix formulations

- Recast the eigenvalue problem for the capacitance matrix C into the Hamiltonian form $i\Phi'(t) = H\Phi(t)$; $H := \sqrt{\frac{\delta}{|D|}} v_r \begin{pmatrix} 0 & \sqrt{C} \\ \sqrt{C} & 0 \end{pmatrix}$.
- Subwavelength eigenmodes need to be almost constant inside the resonators $\Rightarrow$ taking linear combinations of modes would contradict the almost-constant nature of the true modes in the resonators.
- Comparison between a true mode and a tight-binding-type approximant:



Comparison between the tight-binding and capacitance matrix formulations



- Construct tight-binding approximant in the dilute regime: dense ⇒ long-range interactions cannot be ignored and nearest-neighbour approximation cannot be used.

Open questions

- Stability of the resonant frequencies of a system of N resonators under the removal of one resonator or a small number (compared to N) of resonators. See https://royalsocietypublishing.org/doi/full/10.1098/rspa.2021.0765 for the study of the stability properties of graded arrays of subwavelength resonators.
- Retrieve the properties of the surrounding medium from the subwavelength resonant frequencies.
- Optimal design of subwavelength resonator systems. See https://royalsocietypublishing.org/doi/full/10.1098/rspa.2019.0049 for the study of large, graded systems.

Lecture III: Effective medium theory for systems of weakly interacting subwavelength resonators

Large systems of weakly interacting resonators

- Effective medium theory for wave propagation in finite but large systems of weakly interacting subwavelength resonators.
- Below the subwavelength resonant frequency ω_M of a single resonator: high refractive index medium;
- Above ω_M: diffusive medium.
- Dimers of resonators: double negative effective material parameter medium at frequencies slightly higher than the dipole hybridised frequency $\omega_{M,2}$ for a single constituent dimer.
- Subwavelength resonators: ideal building blocks for designing sensors capable of detecting the presence of small particles such as viruses and nanoparticles.
- Measure of the shifts in the structure's resonant frequencies, caused by the perturbations.
- Shift in the resonant frequencies: typically scales in proportion to the size of the perturbation.
- Overcome this weakness through the use of structures with exceptional points.

Large systems of weakly interacting resonators

- Scattering problem:

$$\begin{cases} \Delta u^N + k^2 u^N = 0 \quad \text{in } \mathbb{R}^3 \backslash D^N, \\ \Delta u^N + k_r^2 u^N = 0 \quad \text{in } D^N, \\ u^N|_+ - u^N|_- = 0 \quad \text{on } \partial D^N, \\ \delta \dfrac{\partial u^N}{\partial \nu}\Big|_+ - \dfrac{\partial u^N}{\partial \nu}\Big|_- = 0 \quad \text{on } \partial D^N, \\ u^N - u_{\mathrm{in}} \text{ satisfies the Sommerfeld radiation condition.} \end{cases}$$

- Integral representation of u^N:

$$u^N(x) = \begin{cases} u_{\mathrm{in}} + \mathcal{S}_{D^N}^k[\phi^N], & x \in \mathbb{R}^3 \backslash \overline{D^N}, \\ \mathcal{S}_D^{k_r}[\psi^N], & x \in D^N; \end{cases}$$

- $\phi^N, \psi^N \in L^2(\partial D^N)$:

$$\begin{aligned} \mathcal{S}_{D^N}^k[\phi^N] &= \sum_{1 \leq j \leq N} \mathcal{S}_{D_j^N}^k[\phi_j^N], \\ \mathcal{S}_D^{k_r}[\psi^N] &= \sum_{1 \leq j \leq N} \mathcal{S}_{D_j^N}^{k_r}[\psi_j^N], \end{aligned}$$

with $\phi_j^N, \psi_j^N \in L^2(\partial D_j^N)$.

Scattering problem

- Jump relations $\Rightarrow \phi^N$ and ψ^N:
$$\mathcal{A}^N(\omega,\delta)[\Psi^N] = F^N \quad \text{on } \partial D^N;$$
-
$$\mathcal{A}^N(\omega,\delta) = \begin{pmatrix} \mathcal{S}_{D^N}^{k_r} & -\mathcal{S}_{D^N}^{k} \\ -\frac{1}{2}I + \mathcal{K}_{D^N}^{k_r,*} & -\delta(\frac{1}{2}I + \mathcal{K}_{D^N}^{k,*}) \end{pmatrix};$$
$$\Psi^N = \begin{pmatrix} \psi^N \\ \phi^N \end{pmatrix}, \; F^N = \begin{pmatrix} u_{\text{in}} \\ \delta \frac{\partial u_{\text{in}}}{\partial \nu} \end{pmatrix} \Big|_{\partial D^N}.$$
- $\Rightarrow$ in terms of ϕ_j^N, ψ_j^N:
$$\mathcal{A}_{D_1,\ldots D_N} \begin{pmatrix} \psi_1^N \\ \phi_1^N \\ \vdots \\ \psi_N^N \\ \phi_N^N \end{pmatrix} = \begin{pmatrix} u_{\text{in}}\Big|_{\partial D_1} \\ \frac{\partial u_{\text{in}}}{\partial \nu_1}\Big|_{\partial D_1} \\ \vdots \\ u_{\text{in}}\Big|_{\partial D_N} \\ \frac{\partial u_{\text{in}}}{\partial \nu_N}\Big|_{\partial D_N} \end{pmatrix};$$

ν_i: outward unit normal at $\partial D_i, i = 1, \ldots, N$.

Scattering problem

- $\mathcal{A}_{D_1,\ldots,D_N}$: block diagonal form

$$\mathcal{A}_{D_1,\ldots,D_N} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{L}_{1,2} & \mathcal{L}_{1,3} & \ldots \\ \mathcal{L}_{2,1} & \mathcal{M}_2 & \mathcal{L}_{2,3} & \ldots \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{L}_{N,1} & \mathcal{L}_{N,2} & \ldots & \mathcal{M}_N \end{pmatrix};$$

- $\mathcal{M}_j$: self-interaction for the j^{th} resonator

$$\mathcal{M}_j := \begin{pmatrix} \mathcal{S}_{D_j}^{k_r} & -\mathcal{S}_{D_j}^{k} \\ \frac{1}{\delta}\left(-\frac{1}{2}I + \mathcal{K}_{D_j}^{k_r,*}\right) & -\left(\frac{1}{2}I + \mathcal{K}_{D_j}^{k,*}\right) \end{pmatrix};$$

- $\mathcal{L}_{i,j}, i \neq j$, encodes the effect of the j^{th} resonator j to the i^{th} resonator:

$$\mathcal{L}_{i,j} = \begin{pmatrix} 0 & -\mathcal{S}_{D_i,D_j}^{k} \\ 0 & -\mathcal{L}_{D_i,D_j}^{k} \end{pmatrix}.$$

- $\mathcal{S}_{D_i,D_j}^{k} : L^2(\partial D_j) \to L^2(\partial D_i)$, $\mathcal{L}_{D_i,D_j}^{k} : L^2(\partial D_j) \to L^2(\partial D_i)$:

$$\forall \varphi \in L^2(\partial D_j), \quad \mathcal{S}_{D_i,D_j}^{k}[\varphi] = \mathcal{S}_{D_j}^{k}[\varphi]\Big|_{\partial D_i}, \qquad \mathcal{L}_{D_i,D_j}^{k}[\varphi] = \frac{\partial}{\partial \nu_i}\mathcal{S}_{D_j}^{k}[\varphi]\Big|_{\partial D_i}.$$

Point interaction approximation

- Well-separated resonators $D_j^N = y_j^N + sB$, corresponding subwavelength resonant frequency

$$\omega_M = \frac{1}{s}\sqrt{\frac{\mathrm{Cap}_B \delta}{|B|}} v_r.$$

- $u_j^{\mathrm{in},N}$: field incident on D_j; $u_j^{s,N}$: field scattered from D_j.
-

$$u_i^{\mathrm{in},N}(x) = u_{\mathrm{in}}(x) + \sum_{j\neq i} u_j^{s,N}(x).$$

- Monopole approximation:

$$u_i^{s,N}(x) = g(\omega,\delta,sB) G^k(x - y_i^N) u_i^{\mathrm{in},N}(y_i^N),$$

- $\Rightarrow$ system of linear equations for $u_i^{\mathrm{in},N}(y_i^N)$:

$$u_i^{\mathrm{in},N}(y_i^N) + g(\omega,\delta,sB) \sum_{j\neq i} G^k(y_i^N - y_j^N) u_j^{\mathrm{in},N} = u_{\mathrm{in}}(y_i^N);$$

$$M \begin{pmatrix} u_1^{\mathrm{in},N}(y_1^N) \\ \vdots \\ u_N^{\mathrm{in},N}(y_N^N) \end{pmatrix} = \begin{pmatrix} u_{\mathrm{in}}(y_1^N) \\ \vdots \\ u_{\mathrm{in}}(y_N^N) \end{pmatrix};$$

Large systems of weakly interacting resonators

- M:

$$M_{ij} = \begin{cases} 1, & i = j, \\ g(\omega, \delta, sB) G^k(y_i^N - y_j^N), & i \neq j. \end{cases}$$

- u^N: sum of the incoming wave and all the waves scattered by the different resonators $\Rightarrow$ point interaction approximation of u^N

$$u^N(x) = u_{\mathrm{in}}(x) + \sum_{1 \leq i \leq N} g(\omega, \delta, sB) G^k(x - y_i^N) \sum_{1 \leq j \leq N} (M^{-1})_{ij} u_{\mathrm{in}}(y_j^N).$$

Large systems of weakly interacting resonators

- (H1): $\omega = \mathcal{O}(1)$, independent of N; $1 - (\frac{\omega_M}{\omega})^2 = \beta_0$; for some small constant β_0.
- (H2): $sN = \Lambda$; Λ: positive constant independent of N.
- (H3):

$$\begin{cases} \min_{l \neq j} |y_l^N - y_j^N| \gtrsim N^{-\frac{1}{3}}, \\ s \ll N^{-\frac{1}{3}}. \end{cases}$$

- (H4)$\exists \widetilde{V} \in C^1(\overline{\Omega})$ s.t. for any $f \in C^{0,\alpha}(\Omega)$ with $0 < \alpha < 1$,

$$\max_{1 \leq j \leq N} \Big| \frac{1}{N} \sum_{l \neq j} \frac{1}{|y_l^N - y_j^N|} f(y_l^N) - \int_\Omega \frac{1}{|y - y_j^N|} \widetilde{V}(y) f(y) \, \mathrm{d}y \Big| \lesssim \frac{1}{N^{\frac{\alpha}{3}}} |f\|_{C^{0,\alpha}}.$$

- (H1): deviation of the incident frequency from ω_M; (H2): resonator volume fraction; $N \to +\infty, s, \delta = O(s^2) \to 0$; (H3): size much smaller than the separating distance; (H4): regularity of the sampling points; (H3)+(H4): hold for the periodic distribution.

Effective potential

- $\Omega_N = \Omega \setminus \cup_{1\leq i\leq N} B(y_i^N, \sqrt{s})$;
- There exists some macroscopic field $u \in C^{1,\alpha}(\Omega)$ s.t.

$$u^N(x) \to u(x) \quad \text{for } x \in \Omega_N := \Omega \setminus \cup_{1\leq i\leq N} B(y_i^N, \sqrt{s}).$$

- For any $\epsilon > 0$, there exists N_0 such that for all $N \geq N_0$,

$$\|u^N - u\|_{C^{1,\alpha}(\Omega_N)} \leq \epsilon.$$

- $\Rightarrow$

$$u_j^{\mathrm{in},N}(y_j^N) \to u(y_j^N).$$

- $\Rightarrow$

$$g(\omega, \delta, sB) G^k(x - y_j^N) u_j^{\mathrm{in},N}(y_j^N) \to \frac{1}{N} \Lambda g(\omega, \delta, B) G^k(x - y_j^N) u(y_j^N).$$

Effective potential

- For $x \in \Omega_N$,

$$u(x) = u_{\rm in}(x) + \sum_{1 \le j \le N} g(\omega, \delta, sB) G^k(x - y_j^N) u_j^{{\rm in},N}(y_j^N) + o(1).$$

- $N \to +\infty \Rightarrow$ Lippmann-Schwinger equation:

$$u(x) = u_{\rm in}(x) + \frac{\Lambda {\rm Cap}_B}{\beta_0} \int_\Omega \widetilde{V}(y) G^k(x-y) u(y) \, {\rm d}y.$$

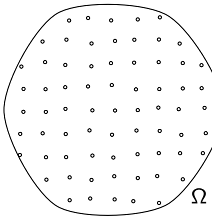
- Applying the operator $\Delta + k^2 \Rightarrow$

$$\Big(\Delta + k^2 - \frac{\Lambda {\rm Cap}_B}{\beta_0} \widetilde{V}(x)\Big) u(x) = 0 \quad \text{in } \Omega.$$

- $-\dfrac{\Lambda {\rm Cap}_B}{\beta_0} \gg 1 \Rightarrow$ effective medium with a high refractive index;
- $-\dfrac{\Lambda {\rm Cap}_B}{\beta_0} \ll -1 \Rightarrow$ diffusive effective medium.

Effective potential

- B: unit sphere; $G^{k_{\mathrm{eff}}}(x)$: Green's function in the presence of the resonators at frequency ω.
- $$g(\omega, \delta, sB) = \frac{4\pi s}{1 - (\frac{\omega_M}{\omega})^2 + \mathrm{i}\gamma_M}, \quad \omega_M = \frac{\sqrt{3\delta}}{s}, \quad \gamma_M = s\omega.$$
- $$\left(\Delta + k^2 - \chi_\Omega \frac{\Lambda \mathrm{Cap}_B}{\beta_0} \widetilde{V}(x)\right) G^{k_{\mathrm{eff}}}(x) = \delta_0 \quad \text{in } \mathbb{R}^3.$$
- Point interaction approximation $\Rightarrow$
$$G^{k_{\mathrm{eff}}}(x) \approx G^k(x) - 4\pi s \sum_{1 \le i \le N} \frac{1}{1 - (\frac{\omega_M}{\omega})^2 + \mathrm{i}s\omega} G^k(x - y_i^N) \sum_j (M^{-1})_{ij} G^k(y_j^N).$$

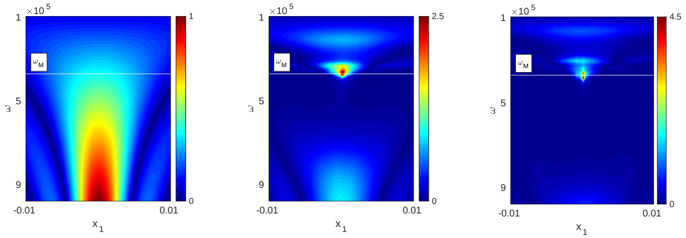


Effective potential

- $\Im(G^{k_{\text{eff}}}(x))$: Point spread function; Resolution: determined by the behavior of the imaginary part of the Green function.
- Helmholtz-Kirchhoff identity:

$$\Im(G^{k_{\text{eff}}}(x)) = k \int_{|y|=R} \overline{G^{k_{\text{eff}}}}(y) G^{k_{\text{eff}}}(x-y)\, d\sigma(y) \quad \text{as } R \to +\infty.$$

- $|\Im(G^{k_{\text{eff}}}(x))|$ for volume fractions $f = 0; 1 \times 10^{-4}; 2 \times 10^{-4}$;
- Sharp peak over the origin just below ω_M: effective refractive index should be greatly enhanced in this frequency regime.



Large systems of weakly interacting dimer resonators

- System of dimers:

$$D^N := \cup_{1\leq j\leq N} D_j^N;$$

- $D_j^N = y_j^N + sR_{d_j^N} D$ for $1 \leq j \leq N$, with y_j^N: center of the dimer D_j^N, s: characteristic size, and $R_{d_j^N}$: rotation in $\mathbb{R}^3$ which aligns the dimer D_j^N in the direction d_j^N, d_j^N: vector of unit length in $\mathbb{R}^3$.
- $0 < s \ll 1$, $N \gg 1$, $\{y_j^N : 1 \leq j \leq N\} \subset \Omega$;
- $(H1)'$: $\delta = \mu^2 s^2$ for some positive number $\mu > 0$, $\omega = \omega_{M,2} + as^2$ for some real number $a \neq \mu^3 \widehat{\eta}_1$;
- (H5): $\exists$ matrix-valued function $\widetilde{B} \in C^1(\overline{\Omega})$ s.t. for any $f \in C^{0,\alpha}(\Omega)$ with $0 < \alpha < 1$,

$$\max_{1\leq j\leq N} \left| \frac{1}{N} \sum_{l\neq j} \Big(d_l^N \cdot \frac{1}{|y_l^N - y_j^N|}\Big) d_l^N \cdot f(y_l^N) - \int_\Omega \widetilde{B}(y) \nabla_y \Big(\frac{1}{|y - y_j^N|}\Big) \cdot f(y)\, \mathrm{d}y \right|$$

$$\lesssim \frac{1}{N^{\frac{\alpha}{3}}} \|f\|_{C^{0,\alpha}}.$$

Large systems of weakly interacting dimer resonators

$$\tilde{g}^0 = \frac{2(C_{11} + C_{12})}{1 - \omega_{M,1}^2/\omega_{M,2}^2}, \quad \tilde{g}^1 = \frac{\mu^2 v_r^2}{2|D|\omega_{M,2}(\mu^3 \widehat{\eta}_1 - a)} P^2,$$

$$M_1 = \begin{cases} I & \text{in } \mathbb{R}^3 \setminus \Omega, \\ I - \Lambda \tilde{g}^1 \tilde{B} & \text{in } \Omega, \end{cases}$$

and

$$M_2 = \begin{cases} k^2 & \text{in } \mathbb{R}^3 \setminus \Omega, \\ k^2 - \Lambda \tilde{g}^0 \tilde{V} & \text{in } \Omega. \end{cases}$$

Large systems of weakly interacting dimer resonators

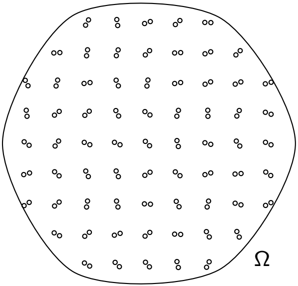
- Suppose that there exists a unique solution u to

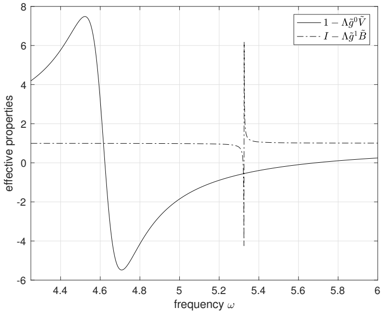
$$\nabla \cdot M_1(x)\nabla u(x) + M_2(x)u(x) = 0 \quad \text{in } \mathbb{R}^3,$$

  s.t. $u - u_{\mathrm{in}}$ satisfies the Sommerfeld radiation condition.
- $\Rightarrow u^N(x) \to u(x)$ uniformly for $x \in \Omega_N$.
- $\tilde{B}$: positive matrix with $\tilde{B}(x) \geq C > 0$ for some constant C for all $x \in \Omega \Rightarrow$ $\omega = \omega_{M,2} + as^2$ with $a < \mu^3 \widehat{\eta}_1$, and sufficiently large Λ, $I - \Lambda \tilde{g}^1 \tilde{B}$ and $k^2 - \Lambda \tilde{g}^0 \tilde{V}$: negative.
- $\Rightarrow$ Effective double-negative medium.
- For $\omega \in [\omega_{M,1}, \omega_{M,2}]$ but away from the dipolar resonance $\omega_{M,2}$, $\tilde{g}^1$ may be small enough s.t. $I - \Lambda \tilde{g}^1 \tilde{B}$: positive, while $k^2 - \Lambda \tilde{g}^0 \tilde{V}$ remains negative $\Rightarrow$ effective medium with one effective single negative material parameter.

Large systems of weakly interacting dimer resonators

- Double-negative effective properties of a system of weakly interacting dimer resonators (D_j^N uniformly distributed on the unit sphere):





Exceptional points in non-Hermitian systems

- $\mathcal{C}$: generalised capacitance matrix. We say that a system of $N \in \mathbb{N}$ resonators $D_1, D_2, \ldots, D_N$ in $\mathbb{R}^3$ admits an N^{th}-order exceptional point if there exists γ s.t.

$$\det(\mathcal{C} - xI) = (\gamma - x)^N,$$
$$\dim \mathrm{Ker}\,(\mathcal{C} - \gamma I) = 1.$$

- Parity–time symmetry: each resonator D_i can be uniquely associated to another resonator D_j (possibly with $i = j$) s.t.

$$D_i = \mathcal{P} D_j, \qquad v_i^2 \delta_i = \mathcal{T}(v_j^2 \delta_j);$$

- Parity operator $\mathcal{P} : \mathbb{R}^3 \to \mathbb{R}^3$; time-reversal operator $\mathcal{T} : \mathbb{C} \to \mathbb{C}$:

$$\mathcal{P}(x) = -x, \qquad \mathcal{T}(z) = \overline{z}.$$

Exceptional points in non-Hermitian systems

- N^{th}-order singularities in $\mathcal{C}$, $\Rightarrow$ design of subwavelength resonant structures with higher-order resonant singularities.
- N^{th}-order exceptional point for $\mathcal{C}$ $\Rightarrow$ there exist N resonant frequencies $\omega_1, \ldots, \omega_N$ and associated eigenmodes $u_1, \ldots, u_N$ s.t. for any $i, j \in \{1, \ldots, N\}$

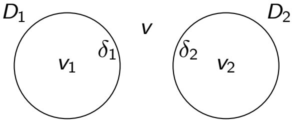
$$\omega_i = \omega_j + \mathcal{O}(\delta), \quad \text{as } \delta \to 0,$$

and for any $i, j \in \{1, \ldots, N\}$ there exists some $K \in \mathbb{C}$ s.t.

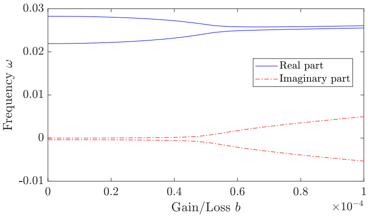
$$u_i = K u_j + \mathcal{O}(\delta), \quad \text{as } \delta \to 0.$$

Exceptional points for PT-symmetric dimers

- Parity-time-symmetric system: $D_1 = -D_2$ and $v_1^2\delta_1 = \overline{v_2^2\delta_2}$



- $v_1^2\delta_1 := a+ib$, $\quad v_2^2\delta_2 := a-ib$, for $a, b \in \mathbb{R}$; $\quad |b|$: magnitude of gain/loss.
- Exceptional points: There is a magnitude of the gain/loss s.t. resonant frequencies and corresponding eigenmodes coincide to leading order in δ.
- $\mathcal{PT}$-symmetry $\Rightarrow$ spectrum of the capacitance matrix to be conjugate symmetric.



Exceptional points for PT-symmetric dimers

- Subwavelength resonant frequencies and corresponding eigenmodes:

$$\omega_i = \omega_i^{(0)} + \mathcal{O}(\delta), \quad u_i = u_i^{(0)} + \mathcal{O}(\delta^{1/2}), \quad \text{as } \delta \to 0,$$

$$\omega_i^{(0)} := \sqrt{\lambda_i}, \quad u_i^{(0)} := v_i^1 S_1^\omega + v_i^2 S_2^\omega.$$

- Eigenvalues of $\mathcal{C}$:

$$\lambda_i = aC_{11} + (-1)^i \sqrt{a^2 C_{12}^2 - b^2 (C_{11}^2 - C_{12}^2)}.$$

- A $\mathcal{PT}$-symmetric pair of subwavelength resonators D_1 and D_2, has an asymptotic exceptional point of order two with respect to δ: There is a set of material parameters s.t. eigenvalues and eigenvectors of the associated generalised capacitance matrix coincide.

- In particular, if

$$\Im(v_1^2 \delta_1) = b^* := \frac{\Re(v_1^2 \delta_1) C_{12}}{\sqrt{C_{11}^2 - C_{12}^2}},$$

then $\lambda_1 = \lambda_2$ and $\mathbf{v}_1 = K \mathbf{v}_2$ for some $K \in \mathbb{C}$, where $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2$: eigenpairs of $\mathcal{C}$.

Exceptional points for PT-symmetric dimers

- If $\Im\left(v_1^2\delta_1\right) < b^*$ then $\sqrt{\lambda_1}, \sqrt{\lambda_2}$ are real valued and $\sqrt{\lambda_1} \neq \sqrt{\lambda_2}$;
- If $\Im\left(v_1^2\delta_1\right) > b^*$ then $\sqrt{\lambda_1}, \sqrt{\lambda_2}$ are purely imaginary and $\sqrt{\lambda_1} \neq \sqrt{\lambda_2}$.

- If $b \neq b^*$, the eigenmodes u_i corresponding to the resonant frequencies ω_i, for $i = 1, 2$:

$$u_i = v_i^1 S_1^\omega + v_i^2 S_2^\omega + \mathcal{O}(\delta^{1/2}),$$

as $\delta \to 0$, where

$$S_j^\omega(x) = \begin{cases} \mathcal{S}_D^k[\psi_j](x), & x \in \mathbb{R}^3 \setminus \overline{D}, \\ \mathcal{S}_D^{k_i}[\psi_j](x), & x \in D_i, i = 1, 2, \end{cases}$$

for $j = 1, 2$, with $v_i = \begin{pmatrix} v_i^1 \\ v_i^2 \end{pmatrix}$ being the eigenvectors of $\mathcal{C}$:

$$v_i = \begin{pmatrix} -C_{12} \\ C_{11} - \mu_i \end{pmatrix}, \qquad \mu_i = \frac{\lambda_i}{(a + \mathrm{i}b)}.$$

Exceptional points in the dilute regime

- Higher-order exceptional points: larger systems of resonators.
- Dilute approximation:

$$D_i = B - \left(i - \frac{N+1}{2} \right) (\epsilon^{-1}, 0, 0).$$

- $\delta := |\delta_1| \ll 1$, $\delta_i = \mathcal{O}(\delta)$, $v_i = \mathcal{O}(1)$ for all $i = 1, \ldots, N$.
- $a \in \mathbb{R}$ s.t. $\Re\left(v_1^2 \delta_1\right) = \delta a$ and assume that $a \neq 0$.
- Define $\mathcal{C}_d^v$ as

$$C_d^v = V \begin{pmatrix} 1 & -\epsilon & -\epsilon/2 & \cdots & -\epsilon/(N-1) \\ -\epsilon & 1 & -\epsilon & \cdots & -\epsilon/(N-2) \\ -\epsilon/2 & -\epsilon & 1 & \cdots & -\epsilon/(N-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\epsilon/(N-1) & -\epsilon/(N-2) & -\epsilon/(N-3) & \cdots & 1 \end{pmatrix}.$$

- V: diagonal; $V_{ii} = \frac{v_i^2 \delta_i}{\delta a}, \quad i = 1, \ldots, N.$

Exceptional points in the dilute regime

- γ_i: eigenvalues of C_d^v and $\underline{q_i}$: corresponding eigenvectors. For small ϵ and δ,

$$\omega_i = \sqrt{\frac{4\pi a\delta\gamma_i}{|D_1|}} + \mathcal{O}(\delta + \delta^{1/2}\epsilon^2),$$

$$u_i(x) = \underline{q_i} \cdot \underline{S}^{\omega_i} + \mathcal{O}(\epsilon^2 + \delta^{1/2}), \quad i = 1, \ldots, N.$$

  Error terms hold uniformly for ϵ and δ in neighbourhoods of 0.

- An N^{th}-order exceptional point of C_d^v: set of parameter values s.t.

$$\det(C_d^v - xI) = (\gamma - x)^N \quad \text{and} \quad \dim \mathrm{Ker}\,(C_d^v - \gamma I) = 1,$$

  for some γ.

- Expand the characteristic polynomial of C_d^v, match the coefficients to those of $(\gamma - x)^N$, and show that the eigenvectors coalesce.

Third-order exceptional point

- Real-valued parameters a, b and c, s.t. $a, b, c = \mathcal{O}(1)$:

$$v_1^2\delta_1 := \delta a(1 + \mathrm{i}b), \qquad v_2^2\delta_2 := \delta ac, \qquad v_3^2\delta_3 := \delta a(1 - \mathrm{i}b),$$

- C_d^v:

$$C_d^v = \begin{pmatrix} 1 + \mathrm{i}b & -(1 + \mathrm{i}b)\epsilon & -(1 + \mathrm{i}b)\epsilon/2 \\ -c\epsilon & c & -c\epsilon \\ -(1 - \mathrm{i}b)\epsilon/2 & -(1 - \mathrm{i}b)\epsilon & 1 - \mathrm{i}b \end{pmatrix}.$$

- Characteristic polynomial $P(x)$ of C_d^v:

$$P(x) = x^3 - (c + 2)x^2 + \left(1 + b^2 + 2c - \frac{\epsilon^2}{4}(1 + b^2 + 8c)\right) x - c(1 + b^2)\left(1 - \frac{9}{4}\epsilon^2 - \epsilon^3\right).$$

- Exceptional point of order 3: $P(x) = (x - \gamma)^3 = x^3 - 3\gamma x^2 + 3\gamma^2 x - \gamma^3$, and $\dim \mathrm{Ker}\,(C_d^v - \gamma I) = 1$.

Third-order exceptional point

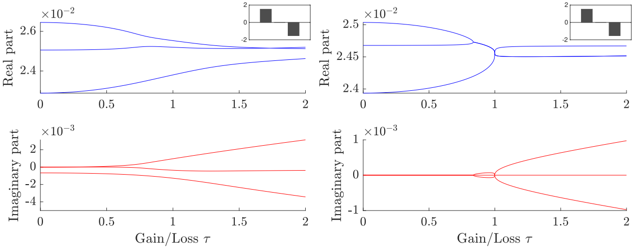
- A $\mathcal{PT}$-symmetric system D of three dilute resonators has an asymptotic exceptional point of order 3 with respect to ϵ and δ at the resonant frequency ω^*, which is given as $\epsilon, \delta \to 0$ by

$$\omega^* = \sqrt{\frac{4\pi(3+\epsilon c_1)\,\Re\left(v_1^2\delta_1\right)}{3|D_1|}} + \mathcal{O}(\delta + \delta^{1/2}\epsilon),$$

where c_1: real root of the polynomial $c_1^3 + \frac{27}{4}c_1 - \frac{27}{8} = 0$ ($c_1 \approx 0.483\ldots$).

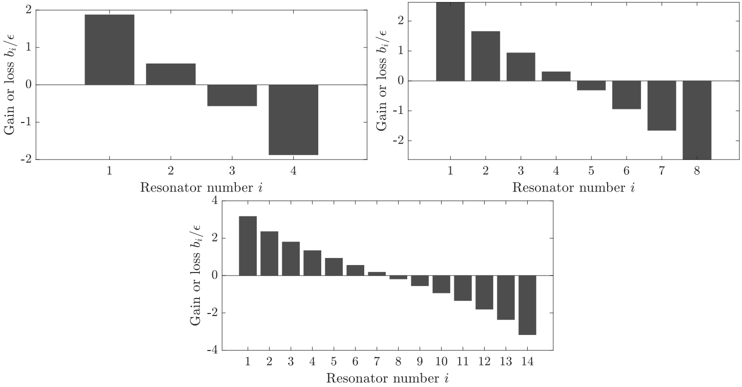
Third-order exceptional point

- A $\mathcal{PT}$-symmetric system of three subwavelength resonators supports an asymptotic exceptional point of order 3; *Left:* resonant frequencies of the full differential problem; *Right:* approximate frequencies using the dilute approximation of the generalised capacitance matrix.



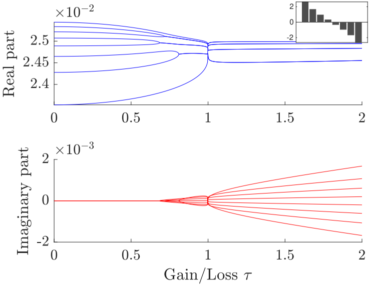
Higher-order exceptional points

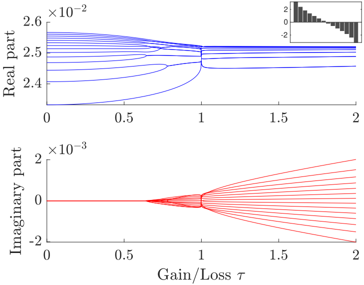
- Higher-order asymptotic exceptional points for $N = 4, 8, 14$:



Higher-order exceptional points

- Patterns of exceptional points of order 8 and 14 as the gain/loss grows linearly:





Open questions

- PT-symmetry at the macroscale:
  - Consider cavities containing many small resonators and use effective medium theory to show that PT symmetry can be replicated at the macroscale;
  - In https://arxiv.org/abs/2003.07796, it is shown that a cavity of resonators with ‘fixed sign’ (i.e., all gain or all loss) converges to an effective system whose material parameters retain this property. It is also observed that a structure that is PT-symmetric at the microscale has real-valued material parameters at the macroscale.
- Stability of the exceptional points with respect to errors in the resonator positions.
- Consider a system of N subwavelength resonators. Tune the material parameters in order to produce exceptional points of order two, three, four, ...
- No exceptional precision of exceptional point sensor ? See https://journals.aps.org/pra/abstract/10.1103/PhysRevA.98.023805.

Lecture IV: Subwavelength bandgap opening, Dirac degeneracies, and resonances in the first radiation continuum

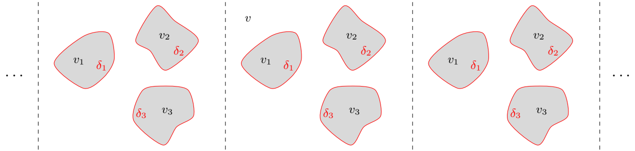
Subwavelength bandgap opening

- Infinite, periodic structure:

$$D_i^m = D_i + m, \qquad \mathcal{D}_i = \bigcup_{m\in\Lambda} D_i + m, \qquad \mathcal{D} = \bigcup_{i=1}^{N} \mathcal{D}_i.$$

- Resonance problem:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^d \setminus \mathcal{D}, \\ \Delta u + k_i^2 u = 0 & \text{in } \mathcal{D}_i, \ i = 1, \ldots, N, \\ u|_+ - u|_- = 0 & \text{on } \partial\mathcal{D}, \\ \delta_i \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial\mathcal{D}_i, \ i = 1, \ldots, N, \\ u(x_l, x_0) & \text{satisfies the outgoing radiation condition as } |x_0| \to \infty. \end{cases}$$



Floquet-Bloch theory

- $f(x) \in L^2(\mathbb{R}^d)$: α-quasiperiodic, with quasiperiodicity $\alpha \in Y^*$, if $e^{-\mathrm{i}\alpha\cdot x} f(x)$ is Λ-periodic.
- Floquet transform of $f \in L^2(\mathbb{R}^d)$:

$$\mathcal{U}[f](x,\alpha) := \sum_{m\in\Lambda} f(x-m)e^{\mathrm{i}\alpha\cdot m}, \quad x,\alpha \in \mathbb{R}^d.$$

- $\mathcal{U}[f]$: α-quasiperiodic in x and periodic in α.
- $\mathcal{U} : L^2(\mathbb{R}^d) \to L^2(Y \times Y^*)$ invertible:

$$\mathcal{U}^{-1}[g](x) = \frac{1}{|Y_l^*|}\int_{Y^*} g(x,\alpha)\,\mathrm{d}\alpha, \quad x \in \mathbb{R}^d,$$

- $g(x,\alpha)$: extended quasiperiodically for x outside of the unit cell Y.

Floquet-Bloch theory

- $u^\alpha(x) := \mathcal{U}[u](x,\alpha)$:

$$\begin{cases} \Delta u^\alpha + k^2 u^\alpha = 0 \quad \text{in } \mathbb{R}^d \setminus \mathcal{D}, \\ \Delta u^\alpha + k_i^2 u^\alpha = 0 \quad \text{in } \mathcal{D}_i, \ i = 1, \dots, N, \\ u^\alpha|_+ - u^\alpha|_- = 0 \quad \text{on } \partial\mathcal{D}, \\ \delta_i \dfrac{\partial u^\alpha}{\partial \nu}\Big|_+ - \dfrac{\partial u^\alpha}{\partial \nu}\Big|_- = 0 \quad \text{on } \partial\mathcal{D}_i, \ i = 1, \dots, N, \\ u^\alpha(x_l, x_0) \text{ is } \alpha\text{-quasiperiodic in } x_l, \\ u^\alpha(x_l, x_0) \text{ satisfies } \alpha\text{-quasiperiodic radiation condition as } |x_0| \to \infty. \end{cases}$$

- Spectrum σ: parameterised by the spectra $\sigma(\alpha)$, $\alpha \in Y^*$, of the Helmholtz resonance problem, which in turn are known to consist of discrete values $\omega = \omega_i^\alpha$:

$$\sigma = \bigcup_{\alpha \in Y^*} \sigma(\alpha), \quad \sigma(\alpha) = \bigcup_{i=1}^{\infty} \omega_i^\alpha.$$

Floquet-Bloch theory

- Band function: $\alpha \mapsto \omega_i^\alpha$; Collection of band functions: band structure.
- Bandgap: connected component of $\mathbb{C} \setminus \sigma$. If σ real, bandgaps of $\mathcal{D}$: intervals in $\mathbb{R}$.
- Subwavelength part of the spectrum: resonant frequencies $\omega_i^\alpha \to 0$ as $\delta \to 0$.

Quasiperiodic layer potentials

- Quasiperiodic Green's function:

$$(\Delta + \omega^2) G^{\alpha,\omega}(x,y) = \sum_{m \in \Lambda} \delta_0(x - y - m) e^{\mathrm{i} m \cdot \alpha} \quad \text{in } \mathbb{R}^d.$$

- Poisson's summation formula:

$$\frac{1}{|Y|} \sum_{q \in \Lambda^*} e^{\mathrm{i}(q+\alpha)\cdot x} = \sum_{m \in \Lambda} \delta_0(x - m) e^{\mathrm{i} m \cdot \alpha}.$$

- If $\omega \neq |q + \alpha|, \forall\, q \in \Lambda^* \Rightarrow$ Spectral representation:

$$G^{\alpha,\omega}(x,y) = \frac{1}{|Y|} \sum_{q \in \Lambda^*} \frac{e^{\mathrm{i}(q+\alpha)\cdot(x-y)}}{\omega^2 - |q+\alpha|^2}.$$

- Spatial representation:

$$G^{\alpha,\omega}(x,y) = \sum_{m \in \Lambda} G^{\omega}(x - m - y) e^{\mathrm{i} m \cdot \alpha}.$$

- Convergence: uniformly for x, y in compact sets of $\mathbb{R}^d$ and $\omega \neq |q + \alpha|$ for all $q \in \Lambda^*$.

Quasiperiodic layer potentials

- For $\varphi \in L^2(\partial D)$,

$$\mathcal{S}_D^{\alpha,\omega}[\varphi](x) = \int_{\partial D} G^{\alpha,\omega}(x,y)\varphi(y)\ \mathrm{d}\sigma(y), \quad x \in \mathbb{R}^d;$$

- Jump relation:

$$\frac{\partial(\mathcal{S}_D^{\alpha,\omega}[\varphi])}{\partial\nu}\bigg|_{\pm}(x) = \left(\pm\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,\omega})^*\right)[\varphi](x) \quad \text{a.e. } x \in \partial D;$$

$$(\mathcal{K}_D^{-\alpha,\omega})^*[\varphi](x) = \int_{\partial D} \frac{\partial G^{\alpha,\omega}(x,y)}{\partial\nu(x)}\varphi(y)\ \mathrm{d}\sigma(y).$$

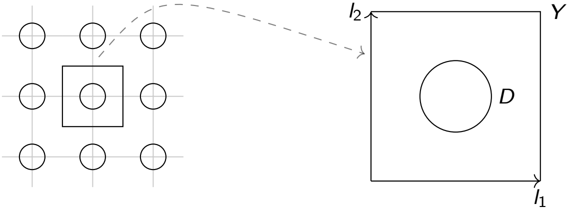
- $\mathcal{S}_D^{\alpha,0} : L^2(\partial D) \to H^1(\partial D)$: invertible for $\alpha \neq 0$.

Subwavelength bandgap opening

- Square lattice crystal: $d_l = d$, $\Lambda = \mathbb{Z}^d$, single resonator $D \in Y = [-1/2, 1/2]^d$;
- For $\alpha \in Y^*$:

$$\left\{\begin{aligned} &\Delta u^\alpha + k^2 u^\alpha = 0 \quad \text{in} \quad Y \backslash \overline{D}, \\ &\Delta u^\alpha + k_r^2 u^\alpha = 0 \quad \text{in} \quad D, \\ &u^\alpha|_+ - u^\alpha|_- = 0 \quad \text{on} \quad \partial D, \\ &\delta \frac{\partial u^\alpha}{\partial \nu}\Big|_+ - \frac{\partial u^\alpha}{\partial \nu}\Big|_- = 0 \quad \text{on} \quad \partial D, \\ &e^{-\mathrm{i}\alpha\cdot x} u^\alpha \ \text{ is periodic.} \end{aligned}\right.$$

- Square lattice crystal:



Subwavelength bandgap opening

- Self-adjoint problem with compact resolvent $\Rightarrow$ nontrivial solutions for discrete values of ω:

$$0 \leq \omega_1^\alpha \leq \omega_2^\alpha \leq \cdots ;$$

- Band structure:

$$[0, \max_\alpha \omega_1^\alpha] \cup [\min_\alpha \omega_2^\alpha, \max_\alpha \omega_2^\alpha] \cup [\min_\alpha \omega_3^\alpha, \max_\alpha \omega_3^\alpha] \cup \cdots .$$

- Representation of Bloch modes:

$$u^\alpha = \begin{cases} \mathcal{S}_D^{\alpha,k}[\phi] & \text{in } Y \setminus \overline{D}, \\ \mathcal{S}_D^{\alpha,k_r}[\psi] & \text{in } D, \end{cases}$$

$\phi, \psi \in L^2(\partial D)$;

- $\Rightarrow \mathcal{A}^\alpha(\omega,\delta)[\Psi] = 0$;

$$\mathcal{A}^\alpha(\omega,\delta) = \begin{pmatrix} \mathcal{S}_D^{\alpha,k_r} & -\mathcal{S}_D^{\alpha,k} \\ -\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,k_r})^* & -\delta(\frac{1}{2} + (\mathcal{K}_D^{-\alpha,k})^*) \end{pmatrix}, \ \Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}.$$

Subwavelength bandgap opening

- $\mathcal{A}^\alpha(\omega,\delta) \in \mathcal{L}(\mathcal{H},\mathcal{H}_1)$; $\mathcal{H} = L^2(\partial D) \times L^2(\partial D)$, $\mathcal{H}_1 = H^1(\partial D) \times L^2(\partial D)$.
- Characteristic values of $\mathcal{A}^\alpha(\omega,\delta)$:

$$0 \le \omega_1^\alpha \le \omega_2^\alpha \le \cdots .$$

- $\mathcal{A}^\alpha(\omega,\delta)$: perturbation of

$$\mathcal{A}^\alpha(\omega,0) = \begin{pmatrix} \mathcal{S}_D^{\alpha,k_r} & -\mathcal{S}_D^{\alpha,k} \\ -\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,k_r})^* & 0 \end{pmatrix}.$$

- ω_0: characteristic value of $\mathcal{A}^\alpha(\omega,0)$ iff $(\omega_0/v_r)^2$: Neumann eigenvalue of D or $(\omega_0/v)^2$: Dirichlet eigenvalue of $Y \setminus D$ with α-quasiperiodic boundary conditions on ∂Y.
- 0: Neumann eigenvalue of $D \Rightarrow \omega_0 = 0$: characteristic value for $\mathcal{A}^\alpha(\omega,0)$.
- Asymptotic Gohberg-Sigal theory $\Rightarrow$ Fix $\alpha \in Y^*$. For any δ sufficiently small, there exists one and only one characteristic value $\omega_1^\alpha(\delta)$ in a neighborhood of the origin in the complex plane to $\mathcal{A}^\alpha(\omega,\delta)$. Moreover, $\omega_1^\alpha(0) = 0$ and ω_1^α depends on δ continuously.

Subwavelength bandgap opening

- For $\alpha \neq 0$ and sufficiently small δ,

$$\omega_1^\alpha = \underbrace{\sqrt{\frac{\delta \mathrm{Cap}_{\alpha,D}}{|D|}} v_r}_{:=\omega_{M,\alpha}} + \mathcal{O}(\delta^{3/2}).$$

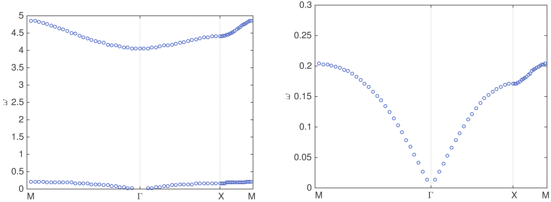
- $\mathrm{Cap}_{\alpha,D} = -\int_{\partial D} (\mathcal{S}_D^{\alpha,0})^{-1}[\chi_{\partial D}]$.
- Formula holds for $d_l = d = 2 \Leftarrow \mathcal{S}_D^{\alpha,0} : L^2(\partial D) \to H^1(\partial D)$: invertible.
- $\omega_{M,\alpha} \to 0 \quad$ as $\alpha \to 0$.
- Dilute regime: $\mathrm{Cap}_{\alpha,D}/\mathrm{Cap}_D \to 1$ for fixed $\alpha \in Y^*$ as the size of D goes to zero.
- $\omega_1^* := \max_\alpha \omega_{M,\alpha}$.
- Subwavelength bandgap opening: For every $\epsilon > 0$, there exists $\delta_0 > 0$ and $\widetilde{\omega} > \omega_1^* + \epsilon$ s.t.

$$[\omega_1^* + \epsilon, \widetilde{\omega}] \subset [\max_\alpha \omega_1^\alpha, \min_\alpha \omega_2^\alpha]$$

  for $\delta < \delta_0$.

Subwavelength bandgap opening

- Band structure of a square array of circular resonators:



High-frequency homogenisation of the Bloch eigenmodes

- D: symmetric with respect to $\{x_j = 0\}$ for $j = 1, \ldots, d \Rightarrow \mathrm{Cap}_{\alpha,D}$ and ω_1^α: attain their maxima at $\alpha^* = (\pi, \ldots, \pi)$.
- For every small $\epsilon > 0$:

$$\mathrm{Cap}_{\alpha^* + \epsilon\tilde{\alpha}, D} = \mathrm{Cap}_{\alpha^*, D} + \epsilon^2 \Lambda_D^{\tilde{\alpha}} + \mathcal{O}(\epsilon^4).$$

- $\Lambda_D^{\tilde{\alpha}}$: negative semidefinite quadratic function of $\tilde{\alpha}$.
- Bloch eigenmode:

$$u_1^\alpha = \mathcal{S}_D^{\alpha, \omega_1^\alpha} \left(\mathcal{S}_D^{\alpha, 0}\right)^{-1} [\chi_{\partial D}] + \mathcal{O}(\delta^{1/2}).$$

High-frequency homogenisation of the Bloch eigenmodes

- Crystal with period s:

$$\omega_{1,s}^{\alpha/s} = \frac{1}{s}\omega_1^{\alpha}, \quad u_{1,s}^{\alpha/s}(x) = u_1^{\alpha}\left(\frac{x}{s}\right).$$

- $\omega^2 - \omega_*^2 = \mathcal{O}(s^2)$:

$$u_{1,s}^{\alpha^*/s+\tilde{\alpha}}(x) = e^{\mathrm{i}\tilde{\alpha}\cdot x} S\left(\frac{x}{s}\right) + \mathcal{O}(s);$$

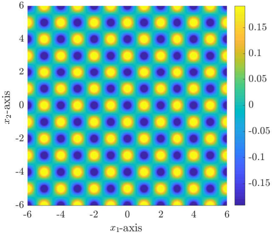
- Macroscopic field $e^{\mathrm{i}\tilde{\alpha}\cdot x}$ satisfies the homogenised equation:

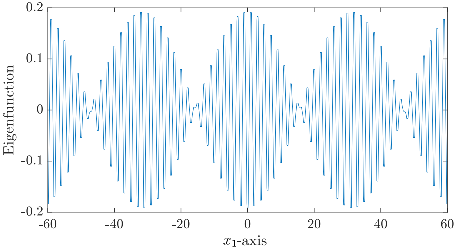
$$\sum_{1\leq i,j\leq d} \lambda_{ij}\partial_i\partial_j \tilde{u}(x) + \frac{\omega_*^2 - \omega^2}{\delta}\tilde{u}(x) = 0;$$

- Microscopic field: periodic and oscillates at the scale of s.

High-frequency homogenisation of the Bloch eigenmodes

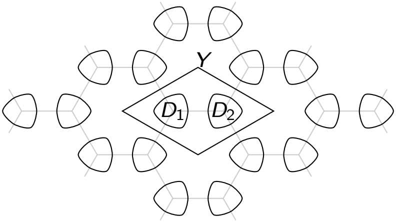
- Real part of Bloch eigenmode of the square lattice:

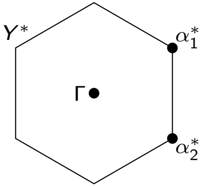


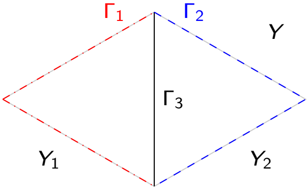


Dirac degeneracies

- Honeycomb crystal and corresponding Brillouin zone Y^*:

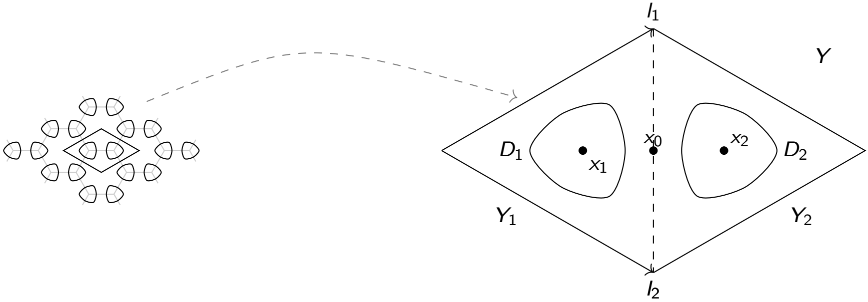






Dirac degeneracies

- Symmetry assumptions:
  - Each resonator: invariant under rotation by $2\pi/3 \Rightarrow D$: invariant under rotation by π; R : rotation by $-2\pi/3$ around the origin; R_1, R_2: rotations by $-2\pi/3$ around x_1 and x_2, respectively; R_0: rotation by π around x_0;
  - $R_1 x = Rx + l_1, \quad R_2 x = Rx + l_2, \quad R_0 x = 2x_0 - x;$



Dirac degeneracies

- Subwavelength band functions $\omega_j^\alpha = \omega_j^\alpha(\delta),\ j = 1, 2$:

$$\omega_j^\alpha = \sqrt{\frac{\delta \lambda_j^\alpha}{|D_1|}} v_r + \mathcal{O}(\delta),$$

uniformly for $\alpha \in Y_0^*$; $|D_1|$: volume of one resonator and $\lambda_j^\alpha,\ j = 1, 2$: eigenvalues of the quasiperiodic capacitance matrix C^α.

- At the Dirac points, C^α : constant multiple of the identity matrix.
- At the Dirac point $\alpha = \alpha^*$ and for δ small enough, the first Bloch resonant frequency $\omega^* := \omega_1^{\alpha^*}$: of multiplicity 2.
- Quasiperiodic capacitance matrix coefficients C_{11}^α and C_{12}^α: differentiable with respect to α at $\alpha = \alpha^*$;

$$\nabla_\alpha C_{11}^\alpha \Big|_{\alpha=\alpha^*} = 0, \quad \nabla_\alpha C_{12}^\alpha \Big|_{\alpha=\alpha^*} = c \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix};$$

$$c := \frac{\partial C_{12}^\alpha}{\partial \alpha_1}\Big|_{\alpha=\alpha^*};$$

- D: symmetric with respect to $\Gamma_3 \Rightarrow c \neq 0$.

Dirac degeneracies

- At $\alpha = \alpha_*$, $\mathcal{C}^\alpha$: eigenvalue of multiplicity 2: $\lambda_1^{\alpha_*} = \lambda_2^{\alpha_*}$;
- Exact degeneracy: For α close to α_* and δ small enough, the first two band functions form a Dirac cone at α_*:

$$\omega_1^\alpha = \omega_* - \mu|\alpha - \alpha_*|\left[1 + \mathcal{O}(|\alpha - \alpha_*|)\right],$$
$$\omega_2^\alpha = \omega_* + \mu|\alpha - \alpha_*|\left[1 + \mathcal{O}(|\alpha - \alpha_*|)\right];$$

- ω_* and μ: independent of α and satisfy

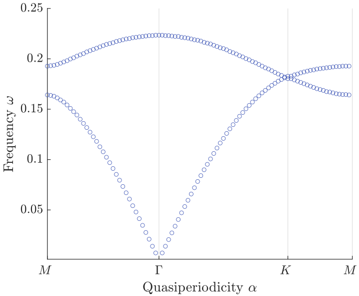
$$\omega_* = \sqrt{\lambda_1^{\alpha_*}} + \mathcal{O}(\delta) \quad \text{and} \quad \mu = |c|\sqrt{\delta}\mu_0 + \mathcal{O}(\delta);$$

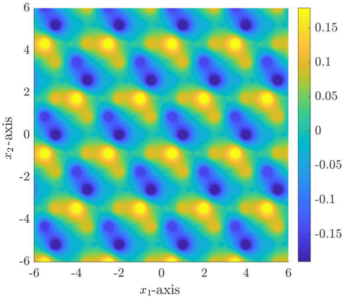
$$\mu_0 = \frac{1}{2}\sqrt{\frac{v_{\mathrm{r}}^2}{|D_1|C_{11}^{\alpha_*}}}, \quad c = \left|\frac{\partial C_{12}^\alpha}{\partial \alpha_1}\Big|_{\alpha=\alpha_*}\right|,$$

  as $\delta \to 0$;
- Error term $\mathcal{O}(|\alpha - \alpha_*|)$: uniform in δ.

Dirac degeneracies

- Dirac cone and small-scale behaviour of the eigenmodes at $\alpha = \alpha^*$:





High-frequency homogenisation

- $\omega - \omega_* = \beta\sqrt{\delta}$:

$$u_s^{\alpha_*/s+\tilde{\alpha}}(x) = \begin{bmatrix} Ae^{\mathrm{i}\tilde{\alpha}\cdot x} \\ Be^{\mathrm{i}\tilde{\alpha}\cdot x} \end{bmatrix} \cdot \mathbf{S}_D^{\alpha_*,k}\left(\frac{x}{s}\right) + \mathcal{O}(s);$$

- Macroscopic field $[\tilde{u}_1, \tilde{u}_2]^\top := [Ae^{\mathrm{i}\tilde{\alpha}\cdot x}, Be^{\mathrm{i}\tilde{\alpha}\cdot x}]^\top$ satisfies the two-dimensional Dirac equation:

$$\mu_0 \begin{bmatrix} 0 & (-c\mathrm{i})(\partial_1 - \mathrm{i}\partial_2) \\ (-\overline{c}\mathrm{i})(\partial_1 + \mathrm{i}\partial_2) & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix} = \frac{\omega - \omega_*}{\sqrt{\delta}} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}.$$

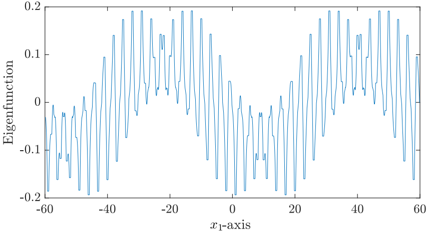
- Each $\tilde{u}_j$ satisfies the Helmholtz equation:

$$\Delta \tilde{u}_j + \frac{(\omega - \omega_*)^2}{\mu^2} \tilde{u}_j = 0.$$

- Zero-phase shift propagation.
- High transmittance $\Leftarrow$ Dirac cone near Γ.

High-frequency homogenisation

- Large-scale behaviour of the eigenmodes close to the Dirac point:



Chains and screens of resonators

- $d_l < d$; Quasiperiodic Green's function $G^{\alpha,k}$ to be solution of

$$(\Delta + k^2)G^{\alpha,k}(x) = \Big(\sum_{m\in\Lambda} \delta_0((x_l,0)-m)e^{\mathrm{i}m\cdot\alpha}\Big)\delta_0(x_0) \quad \text{in } \mathbb{R}^d,$$

  for $x = (x_l, x_0)$ and $\alpha \in Y^*$; radiation condition as $|x_0| \to +\infty$.

- Fourier series expansion + Poisson's summation formula $\Rightarrow k \neq |\alpha + q|$ for all $q \in \Lambda^*$, spectral representation:

$$G^{\alpha,k}(x) = \sum_{q\in\Lambda^*, |\alpha+q|<k} \underbrace{\frac{e^{\mathrm{i}(\alpha+q)\cdot x} e^{\mathrm{i}\sqrt{k^2-|\alpha+q|^2}|x_0|}}{2\mathrm{i}|Y_l|\sqrt{k^2-|\alpha+q|^2}}}_{\text{outgoing modes}} - \sum_{q\in\Lambda^*, |\alpha+q|>k} \underbrace{\frac{e^{\mathrm{i}(\alpha+q)\cdot x} e^{-\sqrt{|\alpha+q|^2-k^2}|x_0|}}{2|Y_l|\sqrt{|\alpha+q|^2-k^2}}}_{\text{evanescent modes}}$$

Chains and screens of resonators

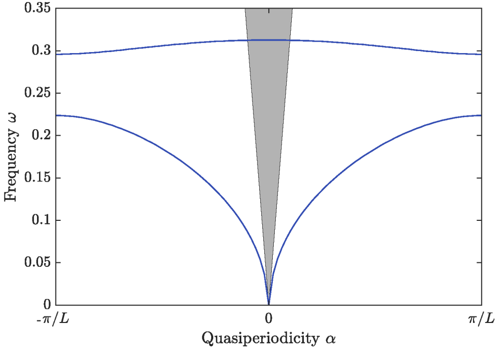
- Quasiperiodic single layer potential $\mathcal{S}_D^{\alpha,k} : L^2(\partial D) \to H^1(\partial D)$: invertible if k is small enough and $k \neq |\alpha + q|$ for all $q \in \Lambda^*$.
- $k < \inf_{q\in\Lambda^*} |\alpha + q|$: exponentially decaying waves away from the structure $\Rightarrow$ evanescent waves;
- $|\alpha| < k < \inf_{q\in\Lambda^*\setminus\{0\}} |\alpha + q|$: propagating waves far away from the structure $\Rightarrow$ first radiation continuum.

Resonances in the first radiation continuum

- Example of the subwavelength band structure of a resonator array with two resonators in the unit cell;
- Shaded region: first radiation continuum first radiation continuum,

$$|\alpha| < \omega/v < \inf_{q \in \Lambda^* \setminus \{0\}} |\alpha + q|;$$

- Unshaded region: evanescent modes.



Chains and screens of resonators

- Quasiperiodic capacitance matrix for $\alpha \neq 0$:

$$C_{ij}^{\alpha} := -\int_{\partial D_i} (\mathcal{S}_D^{\alpha,0})^{-1}[\chi_{\partial D_j}]\, \mathrm{d}\sigma, \quad i,j = 1,\ldots,N;$$

- Equivalently by

$$C_{ij}^{\alpha} := \int_{Y \setminus D} \overline{\nabla V_i^{\alpha}} \cdot \nabla V_j^{\alpha}\, \mathrm{d}x, \quad i,j = 1,\ldots,N;$$

- V_i^{α}, $i = 1,\ldots,N$, solutions

$$\begin{cases} \Delta V_i^{\alpha} = 0 & \text{in } Y \setminus D, \\ V_i^{\alpha} = \delta_{ij} & \text{on } \partial D_j, \\ V_i^{\alpha}(x+l) = e^{\mathrm{i}\alpha\cdot l} V_i^{\alpha}(x) & \forall l \in \Lambda, \\ V_i^{\alpha}(x) \to 0 & \text{as } |x_0| \to \infty, \end{cases}$$

with $x = (x_l, x_0)$.

- C^{α}: Hermitian matrix.
- Generalised quasiperiodic capacitance matrix for $\alpha \neq 0$:

$$\mathcal{C}_{ij}^{\alpha} = \frac{\delta_i v_i^2}{|D_i|} C_{ij}^{\alpha}, \quad i,j = 1,\ldots,N.$$

Resonances in the first radiation continuum

- $|\alpha| \neq 0$ fixed; $\delta \to 0$:

$$\omega_n^\alpha = \sqrt{\lambda_n^\alpha} + \mathcal{O}(\delta^{3/2}), \quad n = 1, \ldots, N;$$

- $\{\lambda_n^\alpha : n = 1, \ldots, N\}$: eigenvalues of $\mathcal{C}^\alpha \in \mathbb{C}^{N\times N}$, which satisfy $\lambda_n^\alpha = \mathcal{O}(\delta)$ as $\delta \to 0$.
- Error term $\mathcal{O}(\delta^{3/2})$: higher order compared to the error term $\mathcal{O}(\delta)$ in the finite case $\Leftarrow \mathcal{O}(\omega)$-term in the expansion of $\mathcal{S}_D^{\alpha,\omega}$ with respect to ω vanishes.
- Resonant modes:

$$u_n^\alpha(x) = \begin{cases} \mathbf{v}_n^\alpha \cdot \mathbf{S}_D^{\alpha,k}(x) + \mathcal{O}(\delta^{1/2}), & x \in \mathbb{R}^d \setminus \overline{\mathcal{D}}, \\ \mathbf{v}_n^\alpha \cdot \mathbf{S}_D^{\alpha,k_i}(x) + \mathcal{O}(\delta^{1/2}), & x \in \mathcal{D}_i, \end{cases}$$

- $k = \omega_n(\alpha)/v$; $k_i = \omega_n(\alpha)/v_i$; $\mathbf{S}_D^{\alpha,k} : \mathbb{R}^d \to \mathbb{C}^N$:

$$\mathbf{S}_D^{\alpha,k}(x) = \begin{pmatrix} \mathcal{S}_D^{\alpha,k}[\psi_1^\alpha](x) \\ \vdots \\ \mathcal{S}_D^{\alpha,k}[\psi_N^\alpha](x) \end{pmatrix}, \quad x \in \mathbb{R}^d \setminus \partial\mathcal{D};$$

- $\psi_i^\alpha := (\mathcal{S}_D^{\alpha,0})^{-1}[\chi_{\partial D_i}]$.

Resonances in the first radiation continuum

- $|\alpha| < k = \omega/v < \inf_{q\in\Lambda^*\setminus\{0\}} |\alpha+q|$:

$$G^{\alpha,k}(x) = \frac{e^{i\alpha\cdot x}e^{ik_0|x_0|}}{2ik_0|Y_l|} - \sum_{q\in\Lambda^*\setminus\{0\}} \frac{e^{i(\alpha+q)\cdot x}e^{-\sqrt{|\alpha+q|^2-k^2}|x_0|}}{2|Y_l|\sqrt{|\alpha+q|^2-k^2}};$$

- $x = (x_l, x_0)$; $k_0 = \sqrt{k^2 - |\alpha|^2}$.
- Periodic (in the x_l-variable) Green's function:

$$G^{0,0}(x) = \frac{|x_0|}{2|Y_l|} - \sum_{q\in\Lambda^*\setminus\{0\}} \frac{e^{iq\cdot x}e^{-|q||x_0|}}{2|Y_l||q|}.$$

- $\omega \to 0$:

$$G^{\omega\alpha_0,k}(x) = \frac{1}{2ik_0|Y_l|} + G^{0,0}(x) + \frac{\alpha\cdot x}{2k_0|Y_l|} + \mathcal{O}(\omega).$$

- $k_0 = \omega\sqrt{1/v^2 - |\alpha_0|^2} \Rightarrow$ Green's function has a singularity when $\omega \to 0$.
- $\widehat{\mathcal{S}}_D^{\alpha,k} : L^2(\partial D) \to H^1(\partial D)$:

$$\widehat{\mathcal{S}}_D^{\alpha,k}[\varphi](x) = \mathcal{S}_D^{0,0}[\varphi](x) - \frac{i-\alpha\cdot x}{2k_0|Y_l|}\int_{\partial D}\varphi(y)\,d\sigma(y) - \int_{\partial D}\frac{\alpha\cdot y}{2k_0|Y_l|}\varphi(y)\,d\sigma(y).$$

- $\omega \to 0$: $\mathcal{S}_D^{\omega\alpha_0,k} = \widehat{\mathcal{S}}_D^{\omega\alpha_0,k} + \omega\mathcal{S}_1^{\alpha_0} + \mathcal{O}(\omega^2)$; $\mathcal{S}_1^{\alpha_0}$: independent of ω.

Resonances in the first radiation continuum

- Dimension of Ker $\mathcal{S}_D^{0,0}$ is at most one; $\mathcal{S}_D^{0,0}$: invertible from the mean-zero space $L^2_0(\partial D)$ onto its image.
- If $\mathcal{S}_D^{0,0}[\varphi] = K$ on ∂D for some constant K and some $\varphi \in L^2(\partial D)$ satisfying $\int_{\partial D} \varphi \, \mathrm{d}\sigma = 0$, then $\varphi = 0$.
- For any $\alpha_0 \in Y^*$ with $0 < |\alpha_0| < 1/v$, $\left(\mathcal{S}_D^{\omega\alpha_0,k}\right)^{-1}$: holomorphic operator-valued function of ω in a neighbourhood of $\omega = 0$.
- $(\mathcal{S}_D^{\omega\alpha_0,k})^{-1}$ does not have the ω^{-1}-singularity around $\omega = 0$.

Resonances in the first radiation continuum

- $$\left(\mathcal{S}_D^{\omega\alpha_0,k}\right)^{-1} = \mathcal{S}_0^{\alpha_0} + \omega\mathcal{S}_{-1}^{\alpha_0} + \mathcal{O}(\omega^2) \quad \text{as } \omega \to 0;$$
- $\mathcal{S}_0^{\alpha_0}$ and $\mathcal{S}_{-1}^{\alpha_0}$: independent of ω.
- For $\alpha \in Y^*$:
$$\psi_i^{\alpha,k} := \left(\widehat{\mathcal{S}}_D^{\alpha,k}\right)^{-1}[\chi_{\partial D_i}].$$
- If $\alpha = \omega\alpha_0$ for some fixed α_0 with $|\alpha_0| < 1/v$:
$$\left(\mathcal{S}_D^{\omega\alpha_0,k}\right)^{-1}[\chi_{\partial D_i}] = \psi_i^0 + \omega\psi_i^{1,\alpha_0} + \mathcal{O}(\omega^2),$$
as $\omega \to 0$, for some $\psi_i^0, \psi_i^{1,\alpha_0} \in L^2(\partial D)$ independent of ω.
- $\psi_i^0 = \mathcal{S}_0^{\alpha_0}[\chi_{\partial D_i}]$; $\psi_i^{1,\alpha_0} = \mathcal{S}_{-1}^{\alpha_0}[\chi_{\partial D_i}]$;
$$\left(\widehat{\mathcal{S}}_D^{\omega\alpha_0,k}\right)^{-1}[\chi_{\partial D_i}] = \psi_i^{\omega\alpha_0,k} = \psi_i^0 + \omega\widehat{\psi}_i^{1,\alpha_0} + \mathcal{O}(\omega^2),$$
as $\omega \to 0$; $\widehat{\psi}_i^{1,\alpha_0} = \psi_i^{1,\alpha_0} + \mathcal{S}_0^{\alpha_0}\mathcal{S}_1^{\alpha_0}[\psi_i^0]$.
- Singular part of $\widehat{\mathcal{S}}_D^{\omega\alpha_0,k}$ must vanish on ψ_i^0:
$$\int_{\partial D} \psi_i^0 \, \mathrm{d}\sigma = 0.$$

Resonances in the first radiation continuum

- Periodic capacitance matrix for α_0 with $|\alpha_0| < 1/v$:

$$C_{ij}^0 = -\int_{\partial D_i} \underbrace{\mathcal{S}_0^{\alpha_0}[\chi_{\partial D_j}]}_{=\psi_j^0} \, \mathrm{d}\sigma, \quad i,j = 1,\ldots,N.$$

-

$$C_{ij}^0 = \int_{Y \setminus D} \nabla V_i^0 \cdot \nabla V_j^0 \, \mathrm{d}x;$$

- V_i^0: unique solution to

$$\begin{cases} \Delta V_i^0 = 0 & \text{in } Y \setminus D, \\ V_i^0 = \delta_{ij} & \text{on } \partial D_j, \\ V_i^0(x_l, x_0) & \text{is } \Lambda\text{-periodic in } x_l, \\ V_i^0(x_l, x_0) \to \pm V_\infty^i & \text{as } x_0 \to \pm\infty; \end{cases}$$

- $V_\infty^i = -\frac{1}{2|Y_l|} \int_{\partial D} y_0 \psi_i^0(y) \, \mathrm{d}\sigma(y), i = 1,\ldots,N$, and may depend on α_0.

Resonances in the first radiation continuum

- Generalised periodic capacitance matrix:

$$\mathcal{C}^0_{ij} = \frac{\delta_i v_i^2}{|D_i|} C^0_{ij}, \quad i,j = 1,\ldots,N.$$

- $\alpha = \omega\alpha_0$ for some α_0 independent of ω and δ s.t. $|\alpha_0| < 1/v$; Helmholtz scattering problem $\Leftrightarrow$ find $\eta \in H^1(\partial D)$ s.t. $\widehat{\mathcal{A}}^\alpha(\omega,\delta)[\eta] = 0$;
- $\widehat{\mathcal{A}}^\alpha(\omega,\delta) : H^1(\partial D) \to L^2(\partial D)$:

$$\widehat{\mathcal{A}}^\alpha(\omega,\delta) = \left(-\frac{1}{2}I + \widetilde{\mathcal{K}}_D^{\omega,*}\right)\left(\widetilde{\mathcal{S}}_D^{\omega}\right)^{-1} - \widetilde{\delta}\left(\frac{1}{2}I + (\mathcal{K}_D^{-\alpha,k})^*\right)\left(\mathcal{S}_D^{\omega\alpha_0,k}\right)^{-1}.$$

- Subwavelength resonant frequencies:

$$\omega_n^0 = \sqrt{\lambda_n^0} + \mathcal{O}(\delta), \quad n = 1,\ldots,N;$$

- $\{\lambda_n^0 : n = 1,\ldots,N\}$: eigenvalues of $\mathcal{C}^0 \in \mathbb{C}^{N\times N}$, which satisfy $\lambda_n^0 = \mathcal{O}(\delta)$ as $\delta \to 0$.

Resonances in the first radiation continuum

- Higher-order approximations:

$$C_{ij}^{1,\alpha_0} = -\int_{\partial D_i} \underbrace{\mathcal{S}_{-1}^{\alpha_0}[\chi_{\partial D_j}]}_{=\psi_j^{1,\alpha_0}} \, \mathrm{d}\sigma;$$

$$\mathcal{C}_{ij}^{1,\alpha_0} = \frac{\delta_i v_i^2}{|D_i|} C_{ij}^{1,\alpha_0}.$$

- $\alpha = \omega\alpha_0$; α_0 independent of ω and δ s.t. $|\alpha_0| < 1/v$. As $\delta \to 0$,

$$\omega_n^0 = \widehat{\omega}_n^0 + \mathcal{O}(\delta^{3/2});$$

- $\widehat{\omega}_n^0$: roots of

$$\det\left(\mathcal{C}^0 + \omega\mathcal{C}^{1,\alpha_0} - \omega^2 I\right) = 0.$$

Resonances in the first radiation continuum

- Modal decomposition:

$$u_n^\alpha(x) = \begin{cases} \mathbf{v}_n^0 \cdot \mathbf{S}_D^{\alpha,k}(x) + \mathcal{O}(\delta^{1/2}), & x \in \mathbb{R}^d \setminus \overline{\mathcal{D}}, \\ \mathbf{v}_n^0 \cdot \mathbf{S}_D^{\alpha,k_i}(x) + \mathcal{O}(\delta^{1/2}), & x \in \mathcal{D}_i; \end{cases}$$

$$\mathbf{S}_D^{\alpha,k}(x) = \begin{pmatrix} \mathcal{S}_D^{\alpha,k}[\psi_1^0 + k\psi_1^{1,\alpha_0}](x) \\ \vdots \\ \mathcal{S}_D^{\alpha,k}[\psi_N^0 + k\psi_N^{1,\alpha_0}](x) \end{pmatrix}, \quad x \in \mathbb{R}^d \setminus \partial\mathcal{D};$$

with $k = \omega_n(\alpha)/v$, $k_i = \omega_n(\alpha)/v_i$, $\psi_i^0 := \mathcal{S}_0^{\alpha_0}[\chi_{\partial D_i}]$; $\psi_i^{1,\alpha_0} := \mathcal{S}_{-1}^{\alpha_0}[\chi_{\partial D_i}]$.

Resonances in the first radiation continuum

- Scattering problem:

$$\begin{cases} \Delta u^\alpha + k^2 u^\alpha = 0 & \text{in } \mathbb{R}^d \setminus \mathcal{D}, \\ \Delta u^\alpha + k_i^2 u^\alpha = 0 & \text{in } \mathcal{D}_i, \ i = 1, \ldots, N, \\ u^\alpha|_+ - u^\alpha|_- = 0 & \text{on } \partial \mathcal{D}, \\ \delta_i \dfrac{\partial u^\alpha}{\partial \nu}\Big|_+ - \dfrac{\partial u^\alpha}{\partial \nu}\Big|_- = 0 & \text{on } \partial \mathcal{D}_i, \ i = 1, \ldots, N, \\ u^\alpha(x_l, x_0) & \text{is } \alpha\text{-quasiperiodic in } x_l, \\ u^\alpha - u_{\mathrm{in}} & \text{satisfies } \alpha\text{-quasiperiodic radiation condition as } |x_0| \to \infty. \end{cases}$$

- $u_{\mathrm{in}}(x) = e^{i\mathbf{k}\cdot x}$; $\alpha = P_l \mathbf{k}$;

$$(u^\alpha - u_{\mathrm{in}})(x) = \sum_{n=1}^{N} a_n u_n^\alpha(x) - \mathcal{S}_D^{\alpha,k} \left(\mathcal{S}_D^{\alpha,k} \right)^{-1} [u_{\mathrm{in}}](x) + \mathcal{O}(\sqrt{\delta});$$

- V: matrix of eigenvectors of $\mathcal{C}^0$; $a_n = a_n(\omega)$ satisfy

$$V \begin{pmatrix} \omega^2 - (\omega_1^0)^2 & & \\ & \ddots & \\ & & \omega^2 - (\omega_N^0)^2 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} \frac{\delta_1 v_1^2}{|D_1|} \int_{\partial D_1} \left(\mathcal{S}_D^{\alpha,k} \right)^{-1} [u_{\mathrm{in}}] \, d\sigma \\ \vdots \\ \frac{\delta_N v_N^2}{|D_N|} \int_{\partial D_N} \left(\mathcal{S}_D^{\alpha,k} \right)^{-1} [u_{\mathrm{in}}] \, d\sigma \end{pmatrix}.$$

Open questions

- Symmetry breaking and bandgap opening in honeycomb structures:
  - Bi-disperse honeycomb lattice: change slightly the radius of one of the two resonators in the unit cell;
  - Perturbed kagome lattice.
- Valley-Hall effect:
- See https://www.nature.com/articles/ncomms16023; https://iopscience.iop.org/article/10.1209/0295-5075/129/44001; https://www.nature.com/articles/s41578-020-0206-0.

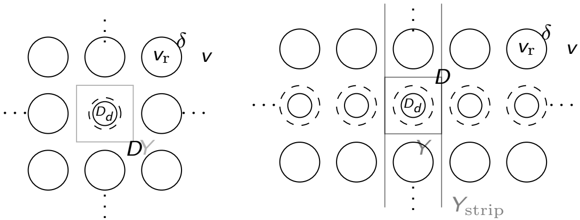
Lecture V: Robust guiding and localisation in Periodic structures at subwavelength scales

Subwavelength guiding of waves

- Helmholtz resonance problem in the fully periodic case $d = d_l = 2$.
- Two defects: either a single resonator or a line of resonators are detuned.
- Square lattice with unit cell $Y = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$;
- D: circle of radius R; D_d: circle of radius $R + \epsilon$ for some $-R < \epsilon < 1 - R$.
- Defect crystals:

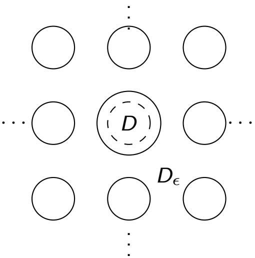
$$\mathcal{D}_{\mathrm{pt}} = \left(\bigcup_{m \in \mathbb{Z}^2 \setminus \{(0,0)\}} D + m \right) \cup D_d;$$

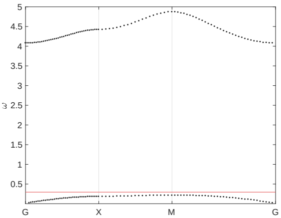
$$\mathcal{D}_{\mathrm{ln}} = \left(\bigcup_{m_1 \in \mathbb{Z}, m_2 \in \mathbb{Z} \setminus \{0\}} D + (m_1, m_2) \right) \cup \left(\bigcup_{m \in \mathbb{Z} \times \{0\}} D_d + m \right).$$



Subwavelength defect modes

- Subwavelength bandgap frequency: if it lies inside a bandgap of the unperturbed structure.
- Defect modes: Create a detuned resonator with an upward shifted resonance frequency (within the subwavelength band gap).
  - Weak interaction $\Rightarrow$ decrease the radius of one resonator (from R to $R+\epsilon$; $\epsilon < 0$);
  - Strong interaction $\Rightarrow$ increase the radius of one resonator (from R to $R+\epsilon$; $\epsilon > 0$);
  - Shift at resonator radius $=$ resonator separation.





Subwavelength defect modes

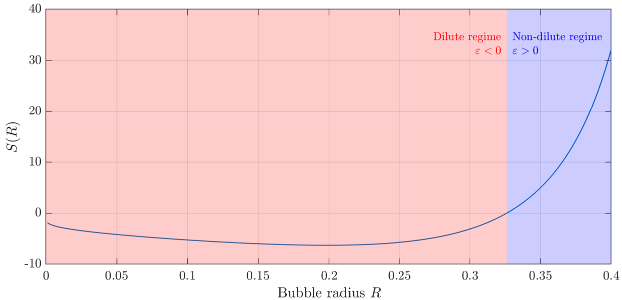
- $$\mathcal{A} := \begin{pmatrix} \mathcal{S}_D^{k_r} & -\mathcal{S}_D^{k} \\ \dfrac{\partial \mathcal{S}_D^{k_r}}{\partial \nu}\Big|_- & -\delta \dfrac{\partial \mathcal{S}_D^{k}}{\partial \nu}\Big|_+ \end{pmatrix} \quad \mathcal{A}_{D_d} := \begin{pmatrix} \mathcal{S}_{D_d}^{k_r} & -\mathcal{S}_{D_d}^{k} \\ \dfrac{\partial \mathcal{S}_{D_d}^{k_r}}{\partial \nu}\Big|_- & -\delta \dfrac{\partial \mathcal{S}_{D_d}^{k}}{\partial \nu}\Big|_+ \end{pmatrix}.$$
- Fictitious source method:
$$\mathcal{P}_1 \begin{pmatrix} e^{\mathrm{i}n\theta} \\ e^{\mathrm{i}m\theta} \end{pmatrix} = \delta_{mn} \frac{R}{R_d} \begin{pmatrix} \dfrac{H_n^{(1)}(k_r R)}{H_n^{(1)}(k_r R_d)} e^{\mathrm{i}n\theta} \\ \dfrac{J_n(kR)}{J_n(kR_d)} e^{\mathrm{i}n\theta} \end{pmatrix}, \quad \mathcal{P}_2 \begin{pmatrix} e^{\mathrm{i}n\theta} \\ e^{\mathrm{i}m\theta} \end{pmatrix} = \delta_{mn} \begin{pmatrix} \dfrac{J_n(kR_d)}{J_n(kR)} e^{\mathrm{i}n\theta} \\ \dfrac{J_n'(kR_d)}{J_n'(kR)} e^{\mathrm{i}n\theta} \end{pmatrix}.$$
- $\mathcal{A}^\epsilon := (\mathcal{P}_2)^{-1} \mathcal{A}_{D_d} \mathcal{P}_1$.
- Subwavelength bandgap frequencies: characteristic values $\omega = \omega^\epsilon(\delta)$ of the operator-valued function
$$\omega \mapsto \mathcal{M}^\epsilon(\omega, \delta) := I + \frac{1}{(2\pi)^2} \left(\mathcal{A}^\epsilon(\omega, \delta) - \mathcal{A}(\omega, \delta) \right) \int_{Y^*} \mathcal{A}^\alpha(\omega, \delta)^{-1} \, \mathrm{d}\alpha$$
inside the subwavelength bandgap of $\mathcal{D}$, s.t. $\omega^\epsilon \to 0$ as $\delta \to 0$.
- $\mathcal{A}^\alpha$: invertible for small enough δ and for ω inside the bandgap,
$$\mathcal{A}^\alpha = \begin{pmatrix} \widetilde{\mathcal{S}}_D^\omega & -\mathcal{S}_D^{\alpha,k} \\ -\frac{1}{2} I + \widetilde{\mathcal{K}}_D^{\omega,*} & -\widetilde{\delta} \left(\frac{1}{2} I + (\mathcal{K}_D^{-\alpha,k})^* \right) \end{pmatrix}.$$

Subwavelength defect modes

- As $\epsilon, \delta \to 0$,

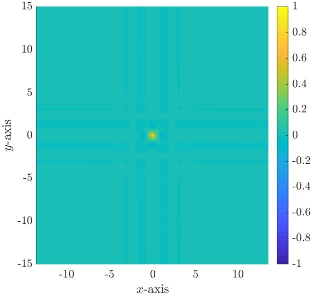
$$\omega^\epsilon - \omega_1^* = \exp\left(-\frac{\mu}{\delta\epsilon} + \mathcal{O}\left(\frac{1}{|\epsilon \ln \delta|}\right)\right);$$

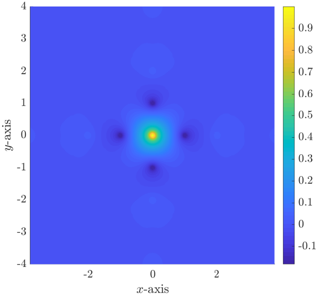
- $\mu = \dfrac{4\pi^2 c_\delta \omega_1^* R^3}{R\|\psi^{\alpha^*}\|^2_{L^2(\partial D)} - 2\mathrm{Cap}_{\alpha^*,D}}$; c_δ: positive constant.
- $S(R) = \left(R\|\psi^{\alpha^*}\|^2_{L^2(\partial D)} - 2\mathrm{Cap}_{\alpha^*,D}\right).$



Subwavelength defect modes

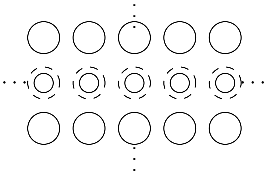
- Real part of the defect eigenmode:

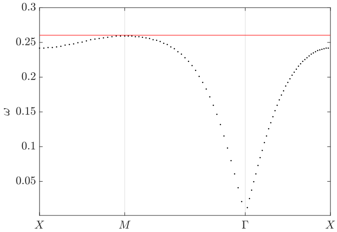


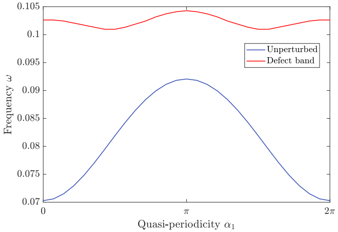


Subwavelength guided modes

- Line defect:
- Defect band within the subwavelength band gap: large perturbation of the radius;
- Defect modes: localised to and guided along the line defect;
- Absence of bound modes.







Subwavelength guided modes

- Subwavelength bandgap frequencies: characteristic values $\omega = \omega^\epsilon(\delta, \alpha_1)$ of the operator-valued function

$$\omega \mapsto \mathcal{M}^{\epsilon,\alpha_1}(\omega,\delta) := I + \frac{1}{2\pi}\left(\mathcal{A}^\epsilon(\omega,\delta) - \mathcal{A}(\omega,\delta)\right)\left(\int_{-\pi}^{\pi} \mathcal{A}^{(\alpha_1,\alpha_2)}(\omega,\delta)^{-1}\, \mathrm{d}\alpha_2\right)$$

  inside the bandgap of $\mathcal{D}$, s.t. $\omega^\epsilon \to 0$ as $\delta \to 0$.

- δ and ϵ: small enough; (R, ϵ) satisfies one of the two assumptions:
  - (i) R is small enough and $\epsilon < 0$ (dilute regime);
  - (ii) R is close enough to $1/2$ and $\epsilon > 0$ (nondilute regime).

- There exists a subwavelength resonant frequency ω^ϵ satisfying $\omega^\epsilon > \omega_1^{\alpha_1,*}$. Moreover, as $\delta, \epsilon \to 0$, we have

$$\omega^\epsilon(\delta,\alpha_1) = \omega_1^{\alpha_1,*}(\delta) + \mu(\alpha_1)\sqrt{\delta}\epsilon^2 + \mathcal{O}\left(\epsilon^2\sqrt{\delta}\left(\frac{1}{|\ln\delta}| + |\epsilon|\right)\right)$$

  for some $\mu = \mu(\alpha_1) > 0$: independent of ϵ and δ.

Subwavelength guided modes

- The whole defect band lies in the subwavelength bandgap:
- For δ and R small enough, and for fixed $\epsilon \in (-R, 0)$, there exists a unique subwavelength resonant frequency ω^ϵ satisfying $\omega^\epsilon > \omega_1^{\alpha_1,*}$. For $\alpha_1 \neq 0$,

$$\omega^\epsilon(\alpha_1) = \widehat{\omega} + \mathcal{O}\left(R^2 + \delta\right),$$

where $\widehat{\omega}$ is the root of the following equation:

$$1 + \frac{1}{2\pi}\left(\frac{\widehat{\omega}^2 R^2}{2\delta} \ln \frac{R}{R_d} + \left(1 - \frac{R^2}{R_d^2}\right)\right) \int_{-\pi}^{\pi} \frac{(\omega^\alpha)^2}{\widehat{\omega}^2 - (\omega^\alpha)^2}\, d\alpha_2 = 0.$$

- For δ and R small enough, and for fixed $\epsilon \in [0, 1 - R)$, there are no resonant frequencies satisfying $\omega^\epsilon > \omega_1^{\alpha_1,*}$.
- For R and δ small enough, there exists an $\epsilon_0 > 0$ s.t. for any $\epsilon \in (-R, -\epsilon_0)$,

$$\omega^\epsilon(\alpha_1) > \omega_1^*$$

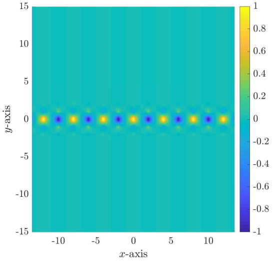
for all $\alpha_1 \in [-\pi, \pi]$.

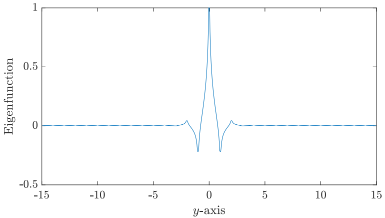
- Defect modes in our case are not bound along the defect line: For δ and R small enough, and for $\alpha_1 \notin \{0, \pi\}$, the subwavelength resonant frequency $\omega^\epsilon = \omega^\epsilon(\alpha_1)$ satisfies

$$\frac{\partial \omega^\epsilon}{\partial \alpha_1} \neq 0.$$

Subwavelength guided modes

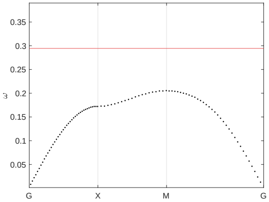
- Real part of the defect eigenmode for $\alpha_1 = \pi/2$ in the dilute case. Each peak corresponds to one resonator, and the defect line is located at $y = 0$:

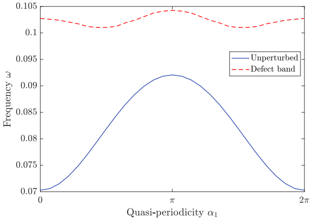




Topological properties of Hermitian systems

- General principle for trapping and guiding waves at subwavelength scales: introduce a defect to a periodic arrangement of subwavelength resonators.
- Sensitivity to imperfections in the crystal's design:

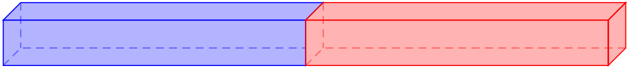




- Goal: design subwavelength wave guides whose properties are robust with respect to imperfections.
- Idea: Topological invariant which captures the crystal's wave propagation properties.
- Topologically protected edge mode.

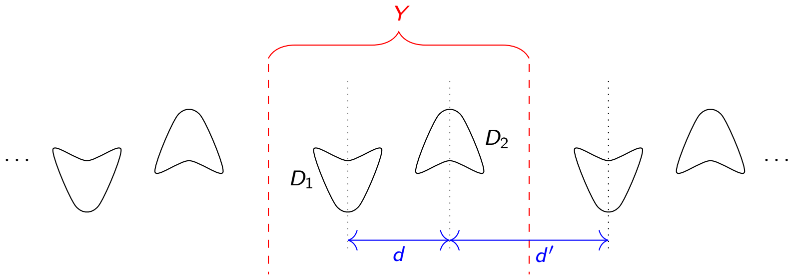
Topological properties of Hermitian systems

- Bulk-boundary correspondence:
  - Take two crystals with topologically different wave propagation properties (different values of the topological invariant);
  - Join half of crystal A to half of crystal B;
  - At the interface, a topologically protected edge mode will exist.



Topological properties of Hermitian systems

- An infinite chain of resonator dimers:[1]



Two assumptions of geometric symmetry:

- dimer is symmetric, in the sense that $D(:= D_1 \cup D_2) = -D$,
- each resonator has reflective symmetry.

[1] Analogue of the Su-Schrieffer-Heeger model in topological insulator theory in quantum mechanics.

Topological properties of Hermitian systems

- The Zak phase:

$$\varphi_n^z := \int_{Y^*} A_n(\alpha)\, d\alpha; \quad Y^* = \mathbb{R}/2\pi\mathbb{Z} \simeq (-\pi, \pi] \quad \text{(first Brillouin zone)};$$

- Berry-Simon connection:

$$A_n(\alpha) := i \int_D u_n^\alpha \frac{\partial}{\partial \alpha} \overline{u}_n^\alpha \, dx; \quad n = 1, 2.$$

- For any $\alpha_1, \alpha_2 \in Y^*$, parallel transport from α_1 to α_2 gives $u_n^{\alpha_1} \mapsto e^{i\theta} u_n^{\alpha_2}$, where θ is given by

$$\theta = \int_{\alpha_1}^{\alpha_2} A_n d\alpha.$$

- $\Rightarrow$ The Zak phase corresponds to parallel transport around the whole of Y^*.

Topological properties of Hermitian systems

- Quasi-periodic capacitance matrix: $C = (C_{ij}^{\alpha})_{i,j=1,2}$.
- The Zak phase is given by the change in the argument of C_{12}^{α} as α varies over the Brillouin zone:
$$\varphi_n^z = -\frac{1}{2}\left[\arg(C_{12}^{\alpha})\right]_{Y^*}.$$
- Further, it holds that
$$C_{12}^{\alpha\,\prime} = e^{-i\alpha} C_{12}^{\alpha}, \Rightarrow \text{if } d = d' \text{then } C_{12}^{\pi} = 0,$$
where the prime denotes that d and d' have been swapped.
- Thus,
$$|\varphi_n^{z\,\prime} - \varphi_n^z| = \pi,$$
i.e. the cases $d > d'$ and $d < d'$ have different Zak phases.

Topological properties of Hermitian systems

- Dilute computations: Assume that the dimer is a rescaling of fixed domains B_1 and B_2:

$$D_1 = \epsilon B_1 - \left(\frac{d}{2}, 0, 0\right), \quad D_2 = \epsilon B_2 + \left(\frac{d}{2}, 0, 0\right),$$

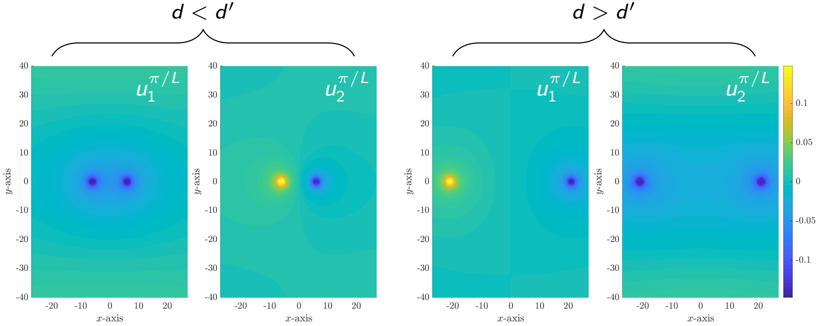
  for $0 < \epsilon$.
- In the dilute regime, as $\epsilon \to 0$:

$$\varphi_n^z = \begin{cases} 0, & \text{if } d < d', \\ \pi, & \text{if } d > d', \end{cases}$$

- There exists a band gap for all $d \neq d'$,
- The dilute crystal has a degeneracy precisely when $d = d'$.
- The dispersion relation has a Dirac cone at $\alpha = \pi$.
- Band inversion occurs between $d < d'$ and $d > d'$.

Topological properties of Hermitian systems

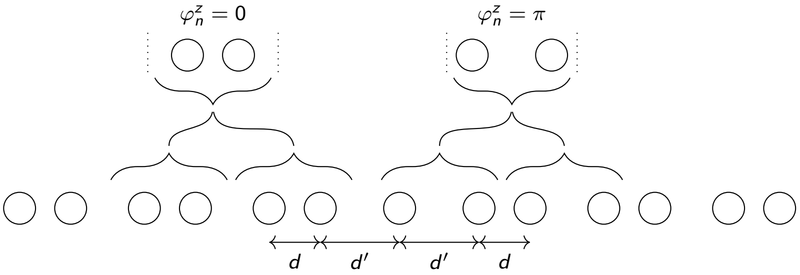
- Band inversion:



The monopole/dipole natures of the 1st and 2nd eigenmodes have swapped between the $d < d'$ and $d > d'$ regimes.

Topological properties of Hermitian systems

- A finite chain of resonators



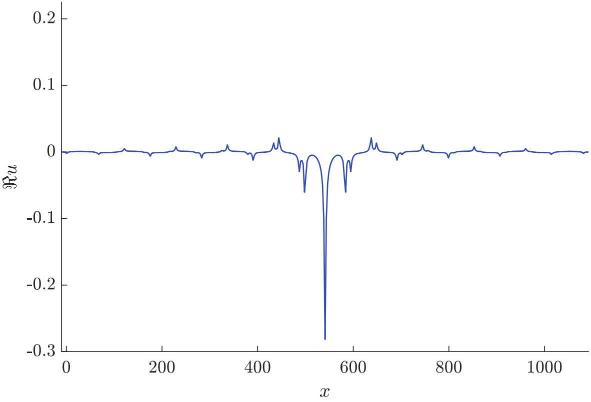
- Capacitance matrix of the finite chain $D = \bigcup_{l=1}^N D_l$:

$$C = (C_{ij}), \quad C_{ij} := -\int_{\partial D_j} (\mathcal{S}_D)^{-1}[\chi_{\partial D_i}], \quad i,j = 1, \ldots, N.$$

- Odd number of resonators $\Rightarrow$ odd number of eigenvalues; middle frequency: midgap frequency $\Rightarrow$ robust to imperfections.

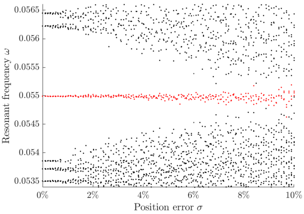
Topological properties of Hermitian systems

- Finite chain - localisation: There is a localised eigenmode

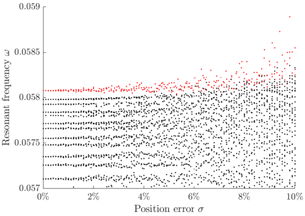


Topological properties of Hermitian systems

- Finite chain–stability to imperfections: Simulation of band gap frequency (red) and bulk frequencies (black) with Gaussian $\mathcal{N}(0, \sigma^2)$ errors added to the resonator positions. σ: expressed as a percentage of the average resonator separation.
- Even for relatively small errors, the frequency associated with the point defect mode exhibits poor stability and is easily lost amongst the bulk frequencies.



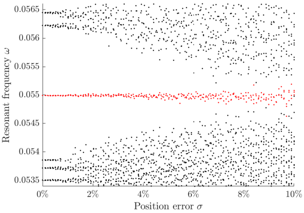
Finite chain with topological interface



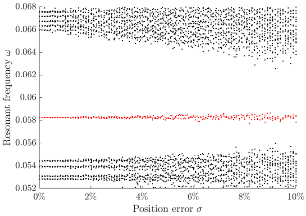
Classical, point defect chain.

Topological properties of Hermitian systems

- Finite chain - effect of diluteness.
- The variance of each frequency is consistent across both dilute and non-dilute regimes.
- In both the dilute and non-dilute regimes, the structure supports a localised mode whose resonant frequency is in the middle of the band gap.
- In the dilute regime, the nearest-neighbour approximation, $C_{ij} = 0$ if $|i - j| > 1$ does not give an accurate approximation $\Rightarrow$ significant difference between classical wave propagation problems and topological insulator theory in quantum mechanics.



Dilute chain, $d = 12,\ d' = 42,\ R = 1$



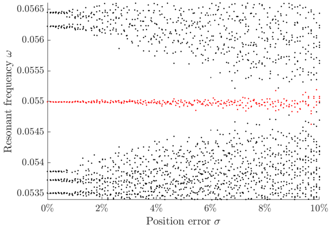
Non-dilute chain, $d = 3,\ d' = 6,\ R = 1$

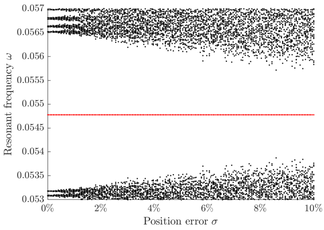
Topological properties of Hermitian systems

- Chiral symmetry: there exists Σ with $\Sigma^2 = I$ s.t. $\widetilde{C}$ satisfies

$$\Sigma \widetilde{C} \Sigma = -\widetilde{C}.$$

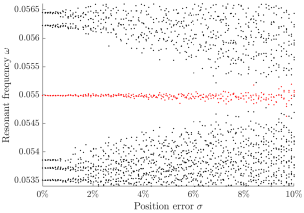
- Chirally symmetric matrix: symmetric spectrum.
- $\widetilde{C}$: subtract the constant diagonal elements from the capacitance matrix C and use a nearest-neighbour approximation.
- $\widetilde{C}$: bisymmetric, tridiagonal matrix with odd size and zero diagonal $\Rightarrow$ chirally symmetric, and has a zero eigenvalue.
- $\widetilde{C}$: retains its chiral symmetry when errors are made in the position of the resonators.



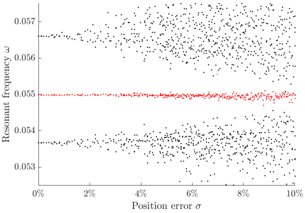


Topological properties of Hermitian systems

- Short finite chains: The stable mode exists also in very short chains of subwavelength resonators.
- With only 9 resonators, there is a midgap frequency which is much more stable than the bulk frequencies.



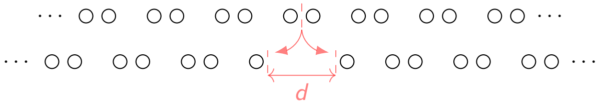
$N = 41$ resonators



$N = 9$ resonators

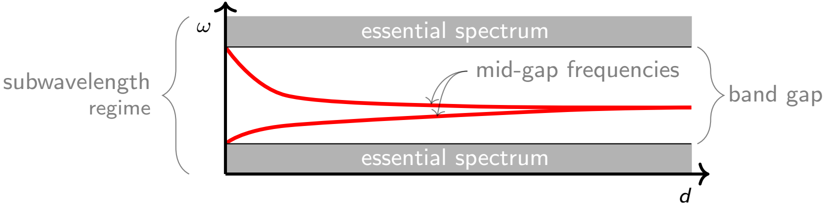
Topological properties of Hermitian systems

- A second approach for creating robust localised subwavelength modes:
  - We start with an array of pairs of subwavelength resonators, known to have a subwavelength band gap. A dislocation (with size $d > 0$) is introduced to create mid-gap frequencies.



Topological properties of Hermitian systems

- As the dislocation size d increases from zero, a mid-gap frequency appears from each edge of the subwavelength band gap. These two frequencies converge to a single value within the subwavelength band gap as $d \to \infty$.



Topological properties of Hermitian systems

- $d \ll 1$: use asymptotic expansions in terms of d to prove that there is a bandgap frequency emerging from each edge of the bandgap;
- $d = mL$ for some $m > 0 \Rightarrow$ dislocation equivalent to removing m dimers from $\mathcal{D}$: explicit computations of the bandgap frequencies in terms of the eigenvalue problem of certain Toeplitz matrices;
- $l_0 = l/L$: ratio of the separation of the resonators to the unit cell length.
- Two fundamentally different cases: $l_0 < 1/2$ and $l_0 > 1/2$.
- First case: dislocation occurs between dimers of resonators, keeping each pair of resonators intact;
- Second case: dislocation occurring within a dimer, splitting one pair of resonators into two "edge" resonators.

Topological properties of Hermitian systems

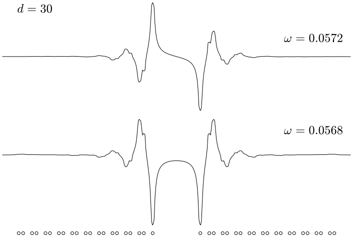
- Assume that D_1 and D_2: strictly convex. For small enough d and δ, and in the case $l_0 > 1/2$, there are two bandgap frequencies $\omega_1(d), \omega_2(d)$ s.t. $\omega_j(d) \to \omega_j^\diamond, j = 1, 2$ as $d \to 0$. In the case $l_0 < 1/2$, there are no bandgap frequencies as $d, \delta \to 0$.
- Assume that the resonators are in the dilute regime and that $l_0 > 1/2$. For small enough δ and ϵ, there exists some $d_0 = \mathcal{O}(\epsilon)$ s.t. there are two bandgap frequencies $\omega_1(d)$ and $\omega_2(d)$ for all $d \in [d_0, +\infty)$, both of which converge to the same value ω_∞ as $d \to +\infty$.
- Bandgap frequencies will cover an interval

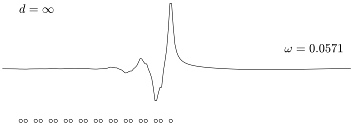
$$\mathcal{I} := [\omega_1(d_0), \omega_2(d_0)]$$

inside the bandgap, and therefore allows us to fine-tune the system to achieve optimal robustness.

Topological properties of Hermitian systems

- Two edge modes for an array of 42 spherical resonators of radius 1; edge mode of the corresponding 'half system':





Bound states in the continuum and Fano resonances

- Localised modes can exist in periodic structures without a defect.
- Symmetries $\Rightarrow$ resonant modes in the radiation continuum whose far field radiation vanishes: bound states in the continuum.
- Resonant mode u_n^α: bound state in the continuum if
  - corresponding resonant frequency ω_n^α: real, satisfies $|\alpha| < \omega_n^\alpha / v$
  - u_n^α satisfies,

$$u_n^\alpha(x_l, x_0) = \mathcal{O}(e^{-K|x_0|}), \quad |x_0| \to +\infty, \quad K > 0.$$

- Subwavelength band structure close to the origin.
- Resonant frequency: real and corresponds to an eigenvalue that is embedded within the continuous radiation spectrum, which is the spectrum of waves that can propagate into the far field.
- Bound state in the continuum: eigenmode associated with this real-valued resonant frequency vanishes in the far field $\Rightarrow$ it will not interact with incoming waves and the corresponding resonance peak will therefore not appear in the transmission spectrum.

Bound states in the continuum and Fano resonances

- Parity operators $\mathcal{P}, \mathcal{P}_0$: $\mathcal{P}(x) = -x$, $\mathcal{P}_0(x_l, x_0) = (x_l, -x_0)$.
- Symmetric screen of dimers repeated periodically:
  - inversion symmetry: $\mathcal{P} D_1 = D_2$;
  - $\mathcal{P}_0 D_i = D_i$ for $i = 1, 2$.
- Inversion symmetry $\Rightarrow$ periodic capacitance matrix C^0 independent of α_0:

$$C^0 = C^0_{11} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

- For $\alpha_0 = 0$, ω_2: real; corresponding mode $u \sim 0$ as $x_0 \to \pm\infty$.

Bound states in the continuum and Fano resonances

- Symmetry broken: the real eigenvalue ω_2 will be shifted into the complex plane and the corresponding mode will be coupled to the far field.
- Design the system so that the two resonances interfere: ω_1 with large imaginary part.
- Derive an expression for the scattering matrix $\Rightarrow$ demonstrate the occurrence of a Fano-type transmission anomaly.
- Existence of asymmetric peaks in transmission spectra due to the interference between a "discrete state" and a "continuum".

Bound states in the continuum and Fano resonances

- At the resonances, $\omega = 0$ or $\omega = \Re(\omega_2)$, scattering matrix corresponding to transmission peaks where transmittance close to 1:

$$S(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mathcal{O}(\delta^{1/2}) \quad \text{and} \quad S(\Re(\omega_2)) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \mathcal{O}(\delta^{1/2});$$

- Widths of the peaks specified by the corresponding imaginary part $\Im(\omega_1), \Im(\omega_2)$.
- Tune the parameters of the system so that $\Im(\omega_1)$: large while $\Im(\omega_2)$: small $\Rightarrow$ for small ω^*,

$$\frac{\omega_1}{\omega_1 - (\Re(\omega_2) - \omega^*)} \approx \frac{\omega_1}{\omega_1 - (\Re(\omega_2) + \omega^*)} \approx \frac{\omega_1}{\omega_1 - \Re(\omega_2)} =: t_1,$$

t_1: not too small.

Bound states in the continuum and Fano resonances

- Transmission coefficient:

$$t(\Re(\omega_2)+\omega^*) \approx \frac{1}{1-\frac{\Re(\omega_2)}{\omega_1}} - \frac{1}{1-\frac{\omega^*}{\mathrm{i}\,\Im(\omega_2)}}.$$

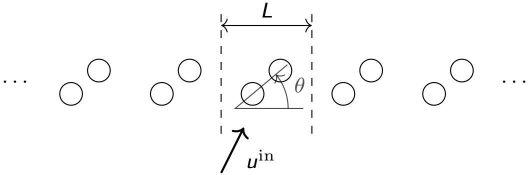
- At $\omega^* = \Re(\omega_2)\frac{\Im(\omega_2)}{\Im(\omega_1)}$,

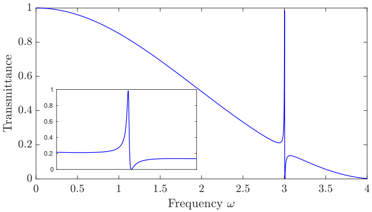
$$t(\Re(\omega_2)+\omega^*) \approx 0, \quad t(\Re(\omega_2)-\omega^*) \approx 2t_1.$$

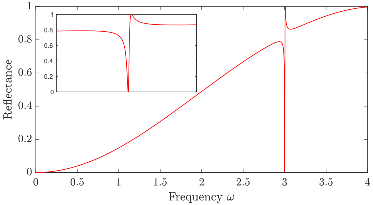
- $\omega^* > 0$; t: close to zero at $\omega = \Re(\omega_2)+\omega^*$ and not at $\omega = \Re(\omega_2)-\omega^* \Rightarrow$ an asymmetric transmission peak at $\omega = \Re(\omega_2)$.
- For some frequency slightly larger than $\Re(\omega_2)$ the transmittance will be close to zero, but for all frequencies slightly lower than $\Re(\omega_2)$ the transmittance will be nonzero.

Bound states in the continuum and Fano resonances

- Resonators arranged in a symmetric dimer that is inclined at an angle of θ to the plane of the screen.







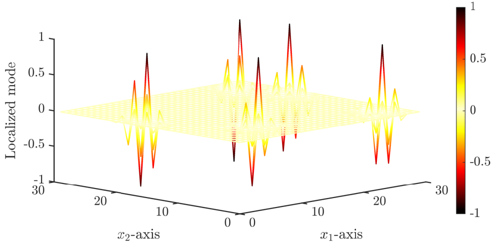
Open questions

- Interface modes in honeycomb structure of subwavelength resonators; zigzag defect; armchair defect; their topological protection; see https://arxiv.org/abs/2405.03238; https://www.nature.com/articles/ncomms16023; https://link.springer.com/article/10.1007/s00205-018-1315-4.
- Topological Valley-Hall interface modes; their Chern numbers; see https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.120.063902.

Lecture VI: Anderson localisation

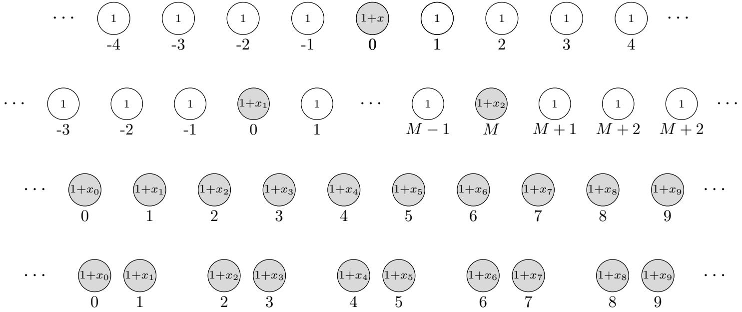
Anderson localisation

- Strong localisation in random media with long-range interactions.
- Scattering of waves by subwavelength resonators with randomly chosen material parameters reproduces the characteristic features of Anderson localisation.
- Hybridisation of subwavelength resonant modes is responsible for both the repulsion of energy levels as well as the phase transition, at which point eigenmode symmetries swap and very strong localisation is possible.
- Characterisation of the localised modes in terms of Laurent operators and generalised capacitance matrices.



Anderson localisation

- Arrays of resonators with defects:



Anderson localisation

- Λ: lattice of dimension $1 \leq d_l \leq d$;

$$\mathcal{D} = \bigcup_{m \in \Lambda} \bigcup_{i \in \{1,\dots,N\}} D_i^m, \qquad D_i^m = D_i + m, \qquad D = \bigcup_{i \in \{1,\dots,N\}} D_i;$$

- v_i^m: wave speed in D_i^m; v: wave speed in the surrounding medium; $\delta_i^m = \mathcal{O}(\delta)$ for $\delta \to 0$.
- Find ω s.t. there exists nontrivial solution u:

$$\begin{cases} \Delta u + \dfrac{\omega^2}{v^2} u = 0 & \text{in } \mathbb{R}^d \setminus \mathcal{D}, \\ \Delta u + \dfrac{\omega^2}{(v_i^m)^2} u = 0 & \text{in } D_i^m, \ i = 1, \dots, N, \\ u|_+ - u|_- = 0 & \text{on } \partial \mathcal{D}, \\ \delta_i^m \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial D_i^m, \ i = 1, \dots, N, \\ u(x_l, x_0) & \text{satisfies an outgoing radiation condition as } |x_0| \to +\infty. \end{cases}$$

Anderson localisation

- Localisation in the L^2-sense:

$$\int_{\mathbb{R}^{d_l}} |u(x_l,0)|^2 \,\mathrm{d}x_l = 1 \quad \text{and} \quad \int_{\mathbb{R}^{d_l}} |u(x_l,x_0)|^2 \,\mathrm{d}x_l < +\infty,$$

for any $x_0 \in \mathbb{R}^{d-d_l}$.

- u: localised, normalised eigenmode corresponding to an eigenvalue ω which satisfies $\omega = \mathcal{O}(\delta^{1/2})$ as $\delta \to 0$. Then, uniformly in $x \in \mathcal{D}$,

$$u(x) = u_i^m + \mathcal{O}(\delta^{1/2}), \quad x \in D_i^m, i = 1, \ldots, N, m \in \Lambda;$$

u_i^m: constant with respect to x and δ.

- Discrete Floquet transform:

$$\mathcal{U}[\phi](\alpha) := \sum_{m\in\Lambda} \phi(m) e^{\mathrm{i}\alpha\cdot m}, \qquad \mathcal{U}^{-1}[\psi](m) := \frac{1}{|Y^*|}\int_{Y^*} \psi(\alpha) e^{-\mathrm{i}\alpha\cdot m}\,\mathrm{d}\alpha.$$

- Real-space capacitance matrix:

$$\widehat{C}^m = \mathcal{U}^{-1}[C^\alpha](m), \quad \widehat{\mathcal{C}}^m = \mathcal{U}^{-1}[\mathcal{C}^\alpha](m), \quad m \in \Lambda.$$

Anderson localisation

- Discrete Floquet transform $\Rightarrow$

$$
\begin{cases}
\Delta u^\alpha + k^2 u^\alpha = 0 & \text{in } Y \setminus \overline{D}, \\
\Delta u^\alpha + \omega^2 \displaystyle\sum_{m \in \Lambda} \frac{e^{\mathrm{i}\alpha\cdot m}}{(v_i^m)^2} u(x + m\Lambda) = 0 & \text{in } D_i, \\
u^\alpha|_+ - u^\alpha|_- = 0 & \text{on } \partial D_i, \\
\delta_i \dfrac{\partial u^\alpha}{\partial \nu}\Big|_+ - \dfrac{\partial u^\alpha}{\partial \nu}\Big|_- = 0 & \text{on } \partial D_i, \\
e^{-\mathrm{i}\alpha\cdot m} u^\alpha(x_l, x_0) & \Lambda\text{-periodic in } x_l, \\
u^\alpha(x_l, x_0) & \text{satisfies an outgoing radiation condition} \\
& \quad \text{as } |x_0| \to +\infty.
\end{cases}
$$

- Inside D_i,

$$
u^\alpha(x) = u_i^\alpha + \mathcal{O}(\delta^{1/2}), \quad x \in D_i, \qquad u_i^\alpha = \sum_{m \in \Lambda} u_i^m e^{\mathrm{i}\alpha\cdot m},
$$

for some sequences $u_i^m \in \ell^2(\mathbb{C})$ for $i = 1, \ldots, N$.

Anderson localisation

- Characterisation of localisation: Any localised solution u corresponding to a subwavelength frequency $\omega = \omega_0 + O(\delta)$, satisfies

$$\mathcal{B}_m \sum_{n \in \Lambda} \mathcal{C}^{m-n} \mathbf{u}^n = \omega_0^2 \mathbf{u}^m,$$

for every $m \in \Lambda$ (real-space variable);

- $\mathcal{C}^m$: inverse Floquet transform of $\mathcal{C}^\alpha$ (real-space capacitance matrix); $\mathbf{u}^m \in \mathbb{R}^N$;
- $\mathcal{B}_m$: $N \times N$ diagonal matrix whose i^{th} entry is given by $b_i^m = 1 + x_i^m$; x_i^m: random perturbation of the material parameter of the resonator i in the cell m.

Laurent-operator formulation

- If $\Lambda = \mathbb{Z}$,

$$\mathfrak{B}\mathfrak{C}\mathfrak{u} = \omega_0^2 \mathfrak{u}.$$

- Doubly infinite matrices and vectors:

$$\mathfrak{C} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \mathcal{C}^0 & \mathcal{C}^1 & \mathcal{C}^2 & \mathcal{C}^3 & \cdots \\ \cdots & \mathcal{C}^{-1} & \mathcal{C}^0 & \mathcal{C}^1 & \mathcal{C}^2 & \cdots \\ \cdots & \mathcal{C}^{-2} & \mathcal{C}^{-1} & \mathcal{C}^0 & \mathcal{C}^1 & \cdots \\ \cdots & \mathcal{C}^{-3} & \mathcal{C}^{-2} & \mathcal{C}^{-1} & \mathcal{C}^0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathfrak{u} = \begin{pmatrix} \vdots \\ \mathbf{u}^{-1} \\ \mathbf{u}^0 \\ \mathbf{u}^1 \\ \mathbf{u}^2 \\ \vdots \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \mathcal{B}_{-1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \mathcal{B}_0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \mathcal{B}_1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \mathcal{B}_2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

- $\mathfrak{C}$: (block) Laurent operator corresponding to the symbol $\mathcal{C}^\alpha$.
- A localised mode corresponds to an eigenvalue of the operator $\mathfrak{B}\mathfrak{C}$.
- In the periodic case (when $\mathfrak{B} = I$), the spectrum of the Laurent operator $\mathfrak{C}$ is continuous and does not contain eigenvalues, so there are no localised modes.
- The operator $\mathfrak{B}\mathfrak{C}$ might have a pure-point spectrum in the non-periodic case.

Anderson's original Hermitian model

- Tight-binding model:

$$H_{\rm tb} = \begin{pmatrix} e_1 & -V & & \\ -V & e_2 & -V & \\ & \ddots & \ddots & \ddots \\ & & -V & e_N \end{pmatrix}, \qquad V > 0.$$

- Disorder supplied by the site energies e_i; independent, uniformly distributed random variables.
- Disorder $\Rightarrow$ entries of $\mathfrak{BC}$: correlated

$$\mathfrak{BC} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \mathcal{B}_{-1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \mathcal{B}_0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \mathcal{B}_1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \mathcal{B}_2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \mathcal{C}^0 & \mathcal{C}^1 & \mathcal{C}^2 & \mathcal{C}^3 & \cdots \\ \cdots & \mathcal{C}^{-1} & \mathcal{C}^0 & \mathcal{C}^1 & \mathcal{C}^2 & \cdots \\ \cdots & \mathcal{C}^{-2} & \mathcal{C}^{-1} & \mathcal{C}^0 & \mathcal{C}^1 & \cdots \\ \cdots & \mathcal{C}^{-3} & \mathcal{C}^{-2} & \mathcal{C}^{-1} & \mathcal{C}^0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Toeplitz matrix formulation for compact defects

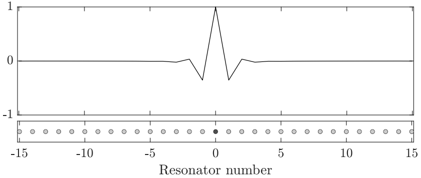
- Compact defects: $\mathcal{B}_m$ are identity for all but finitely many m; $0 \leq m \leq M$.
- X_m: diagonal matrix with entries x_i^m.
- (Block) Toeplitz matrix formulation: ω_0 corresponds to a localised mode iff

$$\det\big(I - \mathcal{X}\mathcal{T}(\omega_0)\big) = 0.$$

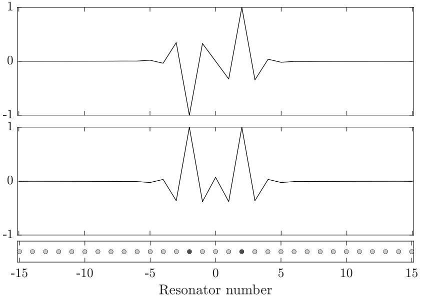
- $\mathcal{X}$: block-diagonal matrix with entries X_m;
- $$\mathcal{T}(\omega) = \begin{pmatrix} T^0 & T^1 & T^2 & \cdots & T^M \\ T^{-1} & T^0 & T^1 & \cdots & T^{M-1} \\ T^{-2} & T^{-1} & T^0 & \cdots & T^{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T^{-M} & T^{-(M-1)} & T^{-(M-2)} & \cdots & T^0 \end{pmatrix};$$
- $$T^m = -\frac{1}{|Y^*|}\int_{Y^*} e^{i\alpha m}\mathcal{C}^\alpha \left(\mathcal{C}^\alpha - \omega^2 I\right)^{-1} d\alpha.$$

Hybridisation and level repulsion

- A single localised mode:

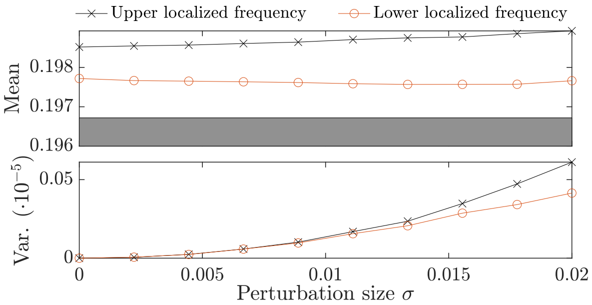


- Two localised modes (higher mode has a dipole (odd) symmetry while the lower mode has a monopole (even) symmetry):



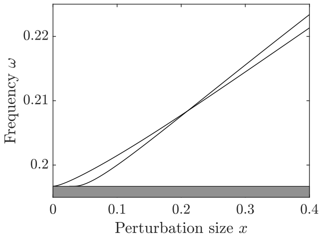
Hybridisation and level repulsion

- The values of x_1 and x_2 are drawn independently from the uniform distribution $U[x - \sqrt{3}\sigma, x + \sqrt{3}\sigma]$.
- Level repulsion: introduction of random perturbations causes the average value of each mid-gap frequency to move further apart (and further apart the edge of the band gap):

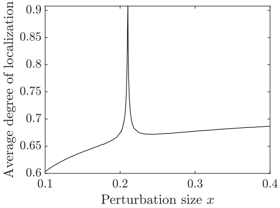


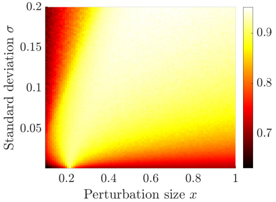
Phase transition and eigenmode symmetry swapping

- Two identical defects of magnitude x;
- Doubly degenerate frequency: a transition point whereby the symmetries of the corresponding eigenmodes swap:



- Sharp peak at the transition point in the degree of localisation:





Open questions

- Topological Fano-resonances: by coupling a continuum mode to a discrete mode that is topologically protected;
- Localisation for k-banded Toeplitz matrices;
- Thouless criterion for localisation/delocalisation;
- Edge mobility; high-frequency homogenisation of the eigenmodes near edge mobility.

Lecture VII: Non-Hermitian, reciprocal periodic systems of subwavelength resonators

Scattering problem

- $D_1, D_2, \ldots, D_N \subset \mathbb{R}^d$, $d \in \{2, 3\}$, $N \in \mathbb{N}$: disjoint, connected sets with boundaries in $C^{1,s}$ for some $0 < s < 1$.
- v_i: wave speed in resonator D_i; $k_i = \omega / v_i$: wave number in D_i, where $\omega \in \mathbb{R}, \omega \neq 0$,: operating frequency; v and k: wave speed and wave number in the background medium.
- Scattering problem:

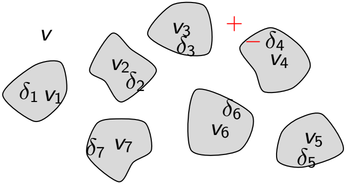
$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^d \setminus \overline{D}, \\ \Delta u + k_i^2 u = 0 & \text{in } D_i, \text{ for } i = 1, \ldots, N, \\ u|_+ - u|_- = 0 & \text{on } \partial D, \\ \delta_i \dfrac{\partial u}{\partial \nu}\Big|_+ - \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial D_i \text{ for } i = 1, \ldots, N, \\ u - u_{\mathrm{in}} \text{ satisfies an outgoing radiation condition.} \end{cases}$$

- High contrast regime $0 < \delta \ll 1$:

$$v, v_i = \mathcal{O}(1), \delta_i = \mathcal{O}(\delta), \qquad \text{for } i = 1, \ldots, N.$$

Subwavelength resonance problem

- Finite collection of resonators:



- Subwavelength resonant frequency: Given $\delta > 0$, a subwavelength resonant frequency $\omega = \omega(\delta) \in \mathbb{C}$:
  - (i) there exists a non-trivial solution to the scattering problem with $u_{\rm in} \equiv 0$, known as an associated resonant mode;
  - (ii) ω depends continuously on δ and satisfies $\omega \to 0$ as $\delta \to 0$.

Capacitance formulation of the resonance problem

- For sufficiently small $\delta > 0$, there exist N subwavelength resonant frequencies $\omega_1(\delta), \ldots, \omega_N(\delta)$ with non-negative real parts.
- C: capacitance matrix

$$C_{ij} = -\int_{\partial D_i} (\mathcal{S}_D^0)^{-1}[\chi_{\partial D_j}] \, \mathrm{d}\sigma, \quad i, j = 1, \ldots, N.$$

- Generalised capacitance matrix:
-

$$\mathcal{C} = V C, \qquad V = \begin{pmatrix} \frac{v_1^2 \delta_1}{|D_1|} & & \\ & \ddots & \\ & & \frac{v_N^2 \delta_N}{|D_N|} \end{pmatrix}$$

Exceptional points in non-Hermitian systems

- $\mathcal{C}$: generalised capacitance matrix. We say that a system of $N \in \mathbb{N}$ resonators $D_1, D_2, \ldots, D_N$ in $\mathbb{R}^3$ admits an N^{th}-order exceptional point if there exists γ s.t.

$$\det(\mathcal{C} - xI) = (\gamma - x)^N,$$
$$\dim \mathrm{Ker}\,(\mathcal{C} - \gamma I) = 1.$$

- Parity–time symmetry: each resonator D_i can be uniquely associated to another resonator D_j (possibly with $i = j$) s.t.

$$D_i = \mathcal{P} D_j, \qquad v_i^2 \delta_i = \mathcal{T}(v_j^2 \delta_j);$$

- Parity operator $\mathcal{P} : \mathbb{R}^3 \to \mathbb{R}^3$; time-reversal operator $\mathcal{T} : \mathbb{C} \to \mathbb{C}$:

$$\mathcal{P}(x) = -x, \qquad \mathcal{T}(z) = \overline{z}.$$

Exceptional points in non-Hermitian systems

- N^{th}-order singularities in $\mathcal{C}$, $\Rightarrow$ design of subwavelength resonant structures with higher-order resonant singularities.
- N^{th}-order exceptional point for $\mathcal{C}$ $\Rightarrow$ there exist N resonant frequencies $\omega_1, \ldots, \omega_N$ and associated eigenmodes $u_1, \ldots, u_N$ s.t. for any $i, j \in \{1, \ldots, N\}$

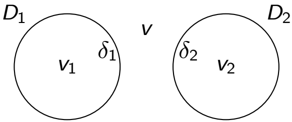
$$\omega_i = \omega_j + \mathcal{O}(\delta), \quad \text{as } \delta \to 0,$$

and for any $i, j \in \{1, \ldots, N\}$ there exists some $K \in \mathbb{C}$ s.t.

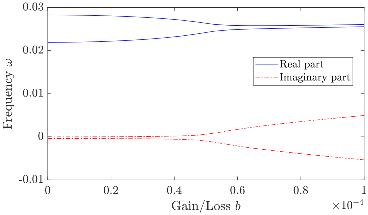
$$u_i = K u_j + \mathcal{O}(\delta), \quad \text{as } \delta \to 0.$$

Exceptional points for PT-symmetric dimers

- Parity-time-symmetric system: $D_1 = -D_2$ and $v_1^2\delta_1 = \overline{v_2^2\delta_2}$



- $v_1^2\delta_1 := a + ib$, $\quad v_2^2\delta_2 := a - ib$, for $a, b \in \mathbb{R}$; $\quad |b|$: magnitude of gain/loss.
- Exceptional points: There is a magnitude of the gain/loss s.t. resonant frequencies and corresponding eigenmodes coincide to leading order in δ.
- $\mathcal{PT}$-symmetry $\Rightarrow$ spectrum of the capacitance matrix to be conjugate symmetric.



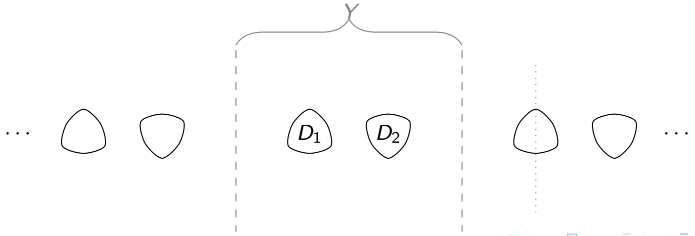
Capacitance matrix of an infinite, periodic system

- Quasiperiodic capacitance matrix for $\alpha \in Y^*, \alpha \neq 0$:

$$C_{ij}^\alpha := \int_{Y \setminus D} \overline{\nabla V_i^\alpha} \cdot \nabla V_j^\alpha \,\mathrm{d}x, \quad i,j = 1,\ldots,N;$$

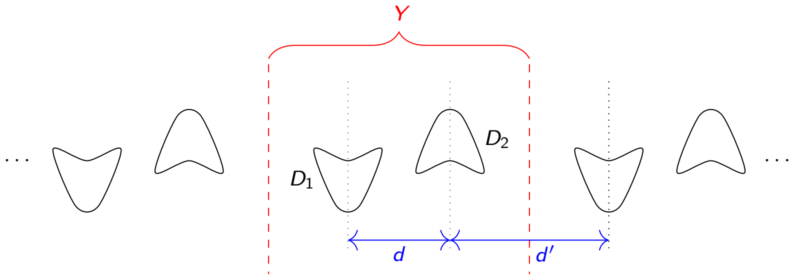
- $$\begin{cases} \Delta V_i^\alpha = 0 & \text{in } Y \setminus \overline{D}, \\ V_i^\alpha = \delta_{ij} & \text{on } \partial D_j, \\ V_i^\alpha(x+l) = e^{\mathrm{i}\alpha\cdot l} V_i^\alpha(x) & \forall l \in \Lambda, \\ V_i^\alpha(x) \to 0 & \text{as } |x_0| \to \infty, \end{cases}$$

  with $x = (x_l, x_0)$.
- C^α : Hermitian; positive definite.



Topological properties of Hermitian systems

- An infinite chain of resonator dimers:



- The Zak phase:

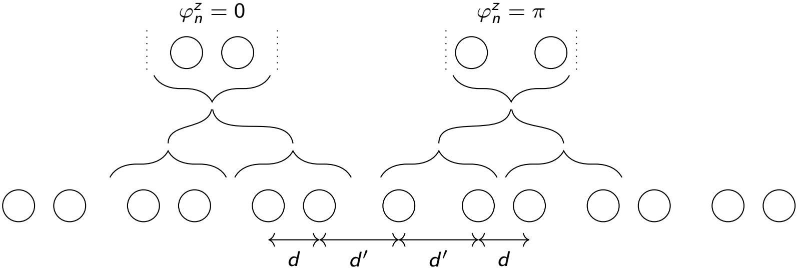
$$\varphi_n^z := i \int_{Y^*} \int_D u_n^\alpha \frac{\partial}{\partial \alpha} \overline{u}_n^\alpha \, dx \, d\alpha;$$

- Given by the change in the argument of C_{12}^α as α varies over the Brillouin zone:

$$\varphi_n^z = -\frac{1}{2} \left[\arg(C_{12}^\alpha)\right]_{Y^*} .$$

Topological properties of Hermitian systems

- A finite chain of resonators



- Capacitance matrix of the finite chain $D = \bigcup_{l=1}^{N} D_l$:

$$C = (C_{ij}), \quad C_{ij} := -\int_{\partial D_j} (\mathcal{S}_D)^{-1}[\chi_{\partial D_i}], \quad i, j = 1, \ldots, N.$$

- Odd number of resonators $\Rightarrow$ odd number of eigenvalues; middle frequency: midgap frequency $\Rightarrow$ robust to imperfections.

Systems with complex material parameters

- Consider dimers in a two-dimensional square lattice with period L in $\mathbb{R}^3$.
- Parity-time symmetry for the dimer $D = D_1 \cup D_2$:

$$\mathcal{P} D_1 = D_2, \quad \delta_1 v_1^2 = \mathcal{T}(\delta_2 v_2^2);$$

  $\mathcal{P}$: parity operator and $\mathcal{T}$: time-reversal operator.

- Consider the regime: $\omega \to 0$ while $|\alpha| > c > 0$ for some c independent of ω.
- Let $v = 1$; $C^\alpha = (C_{ij}^\alpha)_{i,j=1,2}$: quasiperiodic capacitance matrix corresponding to the $\mathcal{PT}$-symmetric metascreen; $\mathcal{C}^\alpha = VC^\alpha$: generalised quasiperiodic capacitance matrix;
- V:

$$V := \begin{pmatrix} \frac{v_1^2 \delta_1}{|D_1|} & 0 \\ 0 & \frac{v_2^2 \delta_2}{|D_2|} \end{pmatrix}.$$

- As $\delta \to 0$, the quasiperiodic resonant frequencies satisfy the asymptotic formula

$$\omega_i^\alpha = \sqrt{\lambda_i^\alpha} + \mathcal{O}(\delta^{3/2}), \quad i = 1, 2,$$

  where λ_i^α: eigenvalues of the generalised quasiperiodic capacitance matrix $\mathcal{C}^\alpha$.

Systems with complex material parameters

- Positive, real-valued parameters a and b:

$$\delta_1 v_1^2 = a + \mathrm{i} b, \quad \delta_2 v_2^2 = a - \mathrm{i} b.$$

- Eigenvalues of $\mathcal{C}$:

$$\lambda_i^\alpha = a C_{11}^\alpha \pm \sqrt{a^2 |C_{12}^\alpha|^2 - b^2 \big((C_{11}^\alpha)^2 - |C_{12}^\alpha|^2 \big)}.$$

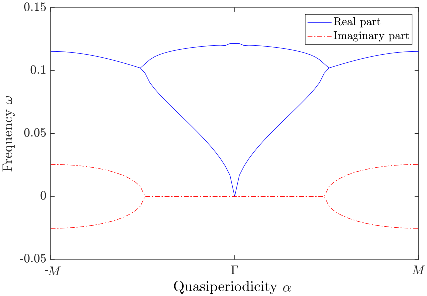
- Exceptional point exceptional point for the $\mathcal{PT}$-symmetric metascreen occurs when $b = b_0(\alpha)$,

$$b_0(\alpha) = \frac{a |C_{12}^\alpha|}{\sqrt{(C_{11}^\alpha)^2 - |C_{12}^\alpha|^2}}.$$

- Exceptional point depends both on the geometry and on the quasiperiodicity α.

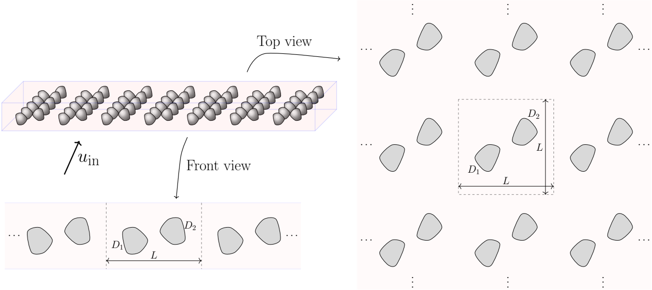
Systems with complex material parameters

- Band structure of a $\mathcal{PT}$-symmetric metascreen:
  - Close to the origin of the Brillouin zone the system is always below the asymptotic exceptional point;
  - For larger α and for large enough b, there will be a point α_0 where $b = b_0(\alpha_0)$;
  - For α above this point, the band structure of the system has a nonzero imaginary part and the two bands are complex-conjugated to leading-order in δ.



Extraordinary transmission and unidirectional reflection

- Metascreen: composed of a $\mathcal{P}$-symmetric resonator dimer $D = D_1 \cup D_2$ repeated periodically in a planar configuration with an incident plane wave $u_{\rm in}$.



Extraordinary transmission and unidirectional reflection

- Extra symmetry condition: $\mathcal{P}_2$ in-plane parity symmetry, $\mathcal{P}_2(x_1, x_2, x_3) = (-x_1, -x_2, x_3)$,

$$\mathcal{P}_2 D_i = D_i, \; i = 1, 2.$$

- Generalised periodic capacitance matrix:

$$\mathcal{C}^0 = VC^0;$$

- Eigenvalues λ_1^0, λ_2^0 and corresponding eigenvectors $\mathbf{v}_1^0, \mathbf{v}_2^0$ of $\mathcal{C}^0$:

$$\lambda_1^0 = 0, \quad \lambda_2^0 = 2aC_{11}^0, \qquad \mathrm{v}_1^0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathrm{v}_2^0 = \begin{pmatrix} -(a + \mathrm{i}b) \\ a - \mathrm{i}b \end{pmatrix}.$$

Extraordinary transmission and unidirectional reflection

- For α_0 s.t. $|\alpha_0| < 1$, the "higher-order" matrix $C^{1,\alpha_0} = (C^{1,\alpha_0})_{i,j=1,2}$:

$$C_{ij}^{1,\alpha_0} = -\frac{\mathrm{i} w_0 L^2}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\mathrm{i} w_0 c_0^2}{2L^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad i,j = 1,2;$$

- $\mathcal{C}^{1,\alpha_0} := VC^{1,\alpha_0}$.
- Second band in the first radiation continuum, $|\alpha| < \omega < \inf_{q \in \Lambda^* \setminus \{0\}} |\alpha + q|$, approximated by

$$\omega_2^\alpha = \sqrt{\frac{2aC_{11}^0}{|D_1|}} + \frac{\mathrm{i} a w_0}{4C_{11}^0}\left(\frac{b^2 L^2}{a^2} - \frac{c_0^2}{L^2}\right) + \mathcal{O}(\delta^{3/2}).$$

Extraordinary transmission and unidirectional reflection

- Incident field u_{in}: sum of plane waves given with wave vectors $\mathbf{k}_{\pm}$;
- ω: real with $0 \leq \omega \leq K\sqrt{\delta}$ for some constant K positive and $\alpha = \omega\alpha_0$. Let $\lambda_2^0, \mathbf{v}_2^0$: the second eigenpair of $\mathcal{C}^0$ and $\lambda = \omega^2$. Assume that

$$\Im(d^\top \mathcal{C}^{1,\alpha_0} \mathbf{v}_2^0) \neq 0, \quad \text{where } d = (1,-1)^\top.$$

Then, for ω s.t. $\lambda = \lambda_2^0 + \lambda^*$, where $\lambda^* = \mathcal{O}(\omega^3)$, the solution to the scattering problem can be written as

$$u - u_{\mathrm{in}} = -(a + \mathrm{i}b)\mu S_1^{\alpha,\omega} + (a - \mathrm{i}b)\mu S_2^{\alpha,\omega} - \mathcal{S}_D^{\alpha,\omega}\left(\mathcal{S}_D^{\alpha,\omega}\right)^{-1}[u_{\mathrm{in}}] + \mathcal{O}(\omega^2);$$

μ given by

$$\mu = \frac{d^\top p}{d^\top\left(\omega \mathcal{C}^{1,\alpha_0} - \lambda^* I\right)\mathrm{v}_2^0} + \mathcal{O}(\omega), \qquad p = -\begin{pmatrix} \frac{v_1^2\delta_1}{|D_1|}\int_{\partial D_1}\left(\mathcal{S}_D^{\alpha,k}\right)^{-1}[u^{\mathrm{in}}]\,\mathrm{d}\sigma \\ \frac{v_2^2\delta_2}{|D_2|}\int_{\partial D_2}\left(\mathcal{S}_D^{\alpha,k}\right)^{-1}[u_{\mathrm{in}}]\,\mathrm{d}\sigma \end{pmatrix}.$$

Error terms: uniform with respect to λ^* in a neighbourhood of 0.

Extraordinary transmission and unidirectional reflection

- Scattering behaviour of the $\mathcal{PT}$-symmetrical screen of resonators:
- D: $\mathcal{PT}$-symmetric; $\mathcal{P}_2 D_i = D_i$ for $i = 1, 2$. Let $\omega_* = \sqrt{\dfrac{2aC_{11}^0}{|D_1|}}$. Assume that $bL^2 \neq ac_0$ and that $\omega \in \mathbb{R}$ s.t. $\omega - \omega_* = \mathcal{O}(\delta)$.
- Asymptotic expansion of the scattering matrix:

$$S(\omega) = \frac{2\mathrm{i}\omega\,\Im\left(\omega_2^\alpha\right)}{(\omega_2^\alpha)^2 - \omega^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{2\omega w_0 b}{a|D_1|\left((\omega_2^\alpha)^2 - \omega^2\right)}\begin{pmatrix} -ac_0 & \mathrm{i}bL^2 \\ \mathrm{i}bL^2 & ac_0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$+\mathcal{O}(\delta^{1/2});$$

Error term: uniform with respect to ω in a neighbourhood of ω_*.

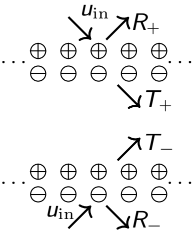
- In particular,

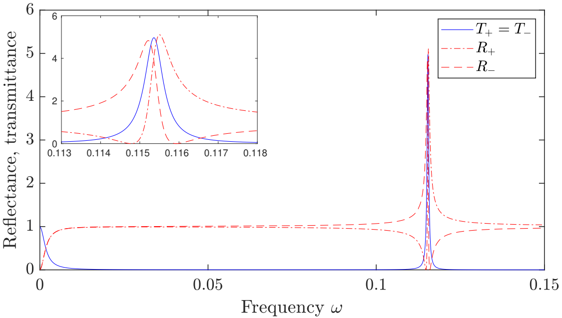
$$r_\pm(\omega) = -\frac{\omega_*^2 - \omega^2 \pm \dfrac{2\omega w_0 b c_0}{|D_1|}}{(\omega_2^\alpha)^2 - \omega^2} + \mathcal{O}(\delta^{1/2});$$

- At leading-order, the reflection coefficients $r_\pm$ vanish at some frequencies $\omega_\pm$.

Extraordinary transmission and unidirectional reflection

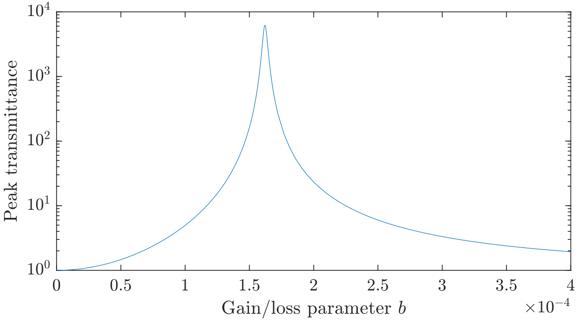
- Transmittances $T_+ = T_-$; Reflectances $R_+ \neq R_-$.
- There are points when $R_\pm$ vanish while $R_\mp$ is nonzero: unidirectional reflection.
- Transmission coefficients: not bounded by unity and can attain large peak values $\Rightarrow$ extraordinary transmission;





Extraordinary transmission and unidirectional reflection

- Extraordinarily high transmittance at $b = ac_0/L^2$:

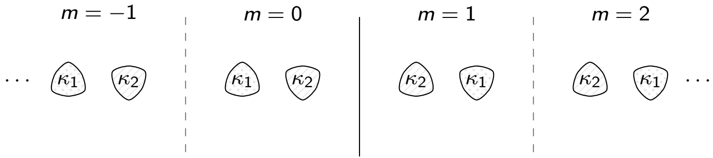


Extraordinary transmission and unidirectional reflection

- In the Hermitian case: real resonances correspond to bound states in the continuum, which decouple from the far field and therefore cannot be excited by incoming waves.
- In the non-Hermitian case: real resonances with modes which are excited by incoming waves.
- Such resonances correspond to extraordinary transmission, where the transmitted field is greatly amplified.
- This amplification, which is impossible in the Hermitian case due to energy conservation, is possible due to the energy input in the non-Hermitian case.

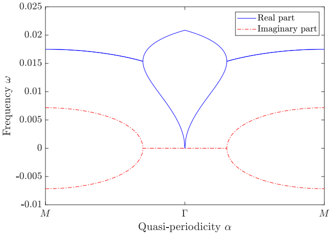
Non-Hermitian band inversion and interface modes

- Localised interface modes in the non-Hermitian case:
  - Localised interface modes in crystals where the periodic geometry is intact, and a defect is placed in the parameters.
  - A topological winding number: the non-Hermitian Zak phase, which describes the winding of the complex eigenvalues.
  - Exceptional point degeneracies can open into non-trivial band gaps enabling non-Hermitian interface modes.



Non-Hermitian band inversion and edge modes

- Exceptional point degeneracy:



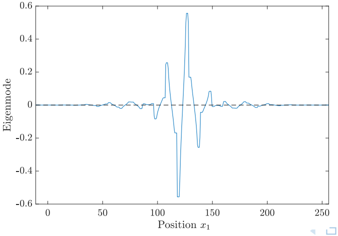
- Non-Hermitian Zak phase: u_j^α: right eigenmode; v_j^α: left eigenmode corresponding to $\overline{\omega_j^\alpha}$,

$$\varphi_j^{\mathrm{zak}} := \frac{i}{2} \int_{Y^*} \left(\left\langle v_j^\alpha, \frac{\partial u_j^\alpha}{\partial \alpha} \right\rangle + \left\langle u_j^\alpha, \frac{\partial v_j^\alpha}{\partial \alpha} \right\rangle \right) d\alpha.$$

- Hermitian counterpart of the structure is topologically trivial.
- Non-Hermitian Zak phase: not quantised but can nevertheless predict the existence of localised interface modes.

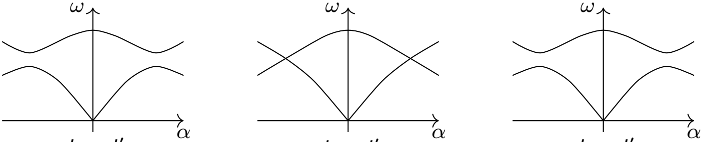
Non-Hermitian band inversion and edge modes

- Exceptional point degeneracy occurs when $\kappa_1 = \overline{\kappa_2} = \kappa$ for sufficiently large κ:
  - $\beta_1 = C_{11}^{\pi} + C_{12}^{\pi}$, $\beta_2 = 2C_{11}^{0}$; $l = (\beta_1 + \beta_2)/(\beta_2 - \beta_1)$.
  - If $\kappa_1 = \overline{\kappa_2} := \kappa$ with $|\mathrm{Im}(\kappa)| \leq \frac{\mathrm{Re}(\kappa)}{\sqrt{l^2-1}}$ (unbroken $\mathcal{PT}$-symmetry), the structure does not support localised modes in the subwavelength regime.
  - If $\kappa_1 = \overline{\kappa_2} := \kappa$ with $|\mathrm{Im}(\kappa)| > \frac{\mathrm{Re}(\kappa)}{\sqrt{l^2-1}}$ (broken $\mathcal{PT}$-symmetry) or if $\kappa_1 \neq \overline{\kappa_2}$ (no $\mathcal{PT}$-symmetry): characterisation of the localised mode in the subwavelength regime.
- Purely non-Hermitian effect: interface modes can be achieved by swapping κ_1 and κ_2 while keeping the distance between the resonators fixed.



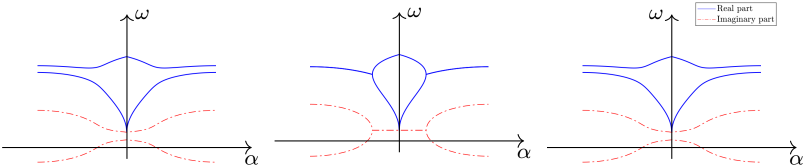
Topological phase transitions

Hermitian:



$d < d'$ No band inversion | $d = d'$ Dirac cone | $d > d'$ Full band inversion

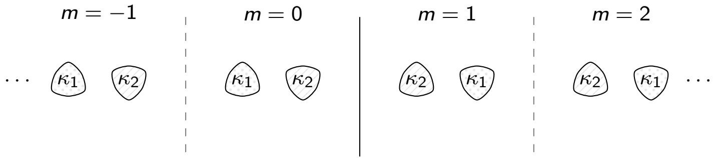
Non-Hermitian:



$|\kappa_1| < |\kappa_2|$ Partial inversion | $\kappa_1 = \overline{\kappa_2}$ Exceptional point | $|\kappa_1| > |\kappa_2|$ Partial inversion

Topological properties of non-Hermitian systems

- Edge modes in the non-Hermitian case:
  - Protected edge modes in crystals where the periodic geometry is intact, and a defect is placed in the parameters.
  - A topological winding number: the non-Hermitian Zak phase, which describes the winding of the complex eigenvalues.
  - Exceptional point degeneracies can open into non-trivial band gaps enabling topologically protected non-Hermitian edge modes.



Open questions

- Exceptional points in honeycomb lattices;
- Non-Hermitian Dirac points
  https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.124.236403.
- When exceptional points meet Dirac singularities:
  https://journals.aps.org/prb/abstract/10.1103/PhysRevB.107.104106.

Lecture IIX: Non-Hermitian, non-reciprocal systems of subwavelength resonators

Non-Hermitian skin effect

- PDE model: $D = \cup_{i=1}^{N}$ chain of finitely many periodic resonators (in x_1-direction) with a non-Hermitian imaginary gauge potential

$$\begin{cases} \Delta u + \omega^2 \dfrac{\rho}{\kappa} u = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D}, \\ \Delta u + \omega^2 \dfrac{\rho_i}{\kappa_i} u + \gamma \partial_{x_1} u = 0 \quad \text{in} \quad D_i, i = 1, \ldots, N, \\ u|_+ = u|_- \quad \text{on} \quad \partial D_i, \\ \dfrac{\rho_i}{\rho} \dfrac{\partial u}{\partial \nu}\Big|_+ = \dfrac{\partial u}{\partial \nu}\Big|_- \quad \text{on} \quad \partial D_i, \\ u \text{ satisfies the radiation condition.} \end{cases}$$

- Condensation of bulk eigenmodes at one of the edges of the system (depending on $\mathrm{sign}(\gamma)$) as its size increases.

Non-Hermitian skin effect

- Green's function:

$$G_\gamma^\omega(x) = -\frac{\exp(-\gamma x_1/2 + \mathrm{i}\sqrt{\omega^2 - \gamma^2/4}|x|)}{4\pi|x|}.$$

- $\Delta G_\gamma^\omega + \omega^2 G_\gamma^\omega + \gamma \partial_{x_1} G_\gamma^\omega = \delta_0$ in $\mathbb{R}^d$.
- Characteristic values:

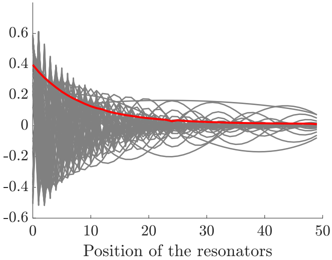
$$\underbrace{\begin{pmatrix} \widetilde{\mathcal{S}}_{\mathcal{D},\gamma}^{\omega} & -\mathcal{S}_{\mathcal{D}}^{\omega} \\ -\frac{1}{2}I + \widetilde{\mathcal{K}}_{\mathcal{D},\gamma}^{\omega,*} & -\delta_i(\frac{1}{2}I + \mathcal{K}_{\mathcal{D}}^{\omega,*}) \end{pmatrix}}_{:=\mathcal{A}(\omega,\delta)} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = 0$$

- Eigenmodes and eigenfrequencies approximated by the eigenvectors and square roots of the eigenvalues of the gauge capacitance matrix:

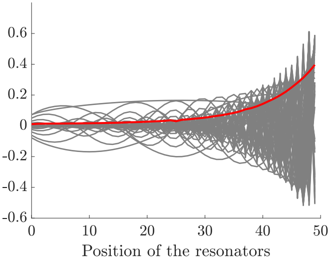
$$(\mathcal{C}_N^\gamma)_{i,j} = -\frac{\delta_i v_i^2}{\int_{D_i} e^{\gamma x_1}\, \mathrm{d}x} \int_{\partial D_i} e^{\gamma x_1} (\mathcal{S}_D^0)^{-1}[\chi_{\partial D_j}] \mathrm{d}\sigma(x).$$

Non-Hermitian skin effect

- Eigenvectors of the gauge capacitance matrix are exponentially decaying or growing, depending on the sign of γ:



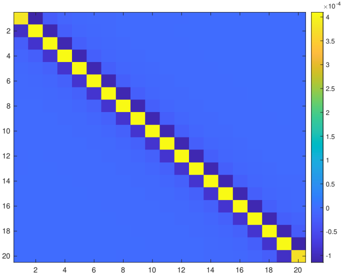
$\gamma = 1.$



$\gamma = -1.$

Non-Hermitian skin effect

- Gauge capacitance matrix $\mathcal{C}^\gamma$: perturbed Toeplitz structure $\Leftarrow$ system: almost translational invariant;
- long-range coupling in three dimensions $\Rightarrow$ $\mathcal{C}^\gamma$: dense.
- $\mathcal{C}^\gamma$: approximated by a banded Toeplitz matrix with a perturbation on the edge.
- Symbol function: $f(z) = \sum_{j=-(k-1)}^{k-1} a_j z^j$.



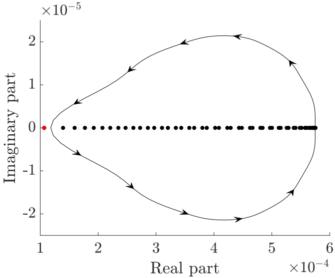
$$\approx \begin{bmatrix} a_0 & \cdots & a_{-k-1} & 0 & \cdots & 0 \\ \vdots & a_0 & \ddots & \ddots & & \vdots \\ a_{k+1} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & a_{-k-1} \\ \vdots & & \ddots & \ddots & a_0 & \vdots \\ 0 & \cdots & 0 & a_{k+1} & \cdots & a_0 \end{bmatrix}$$

Non-Hermitian skin effect

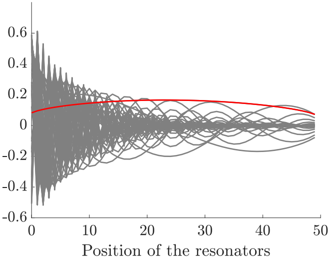
- Define $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $I(f(\mathbb{T}), \lambda)$ the winding number of $f(\mathbb{T})$ at λ in the positive direction.
- Exponential decay of the pseudo-eigenvectors: predicted by the winding number:

$$\frac{\left|\left(\mathbf{v}^{(N)}\right)_j\right|}{\max_j \left|\left(\mathbf{v}^{(N)}\right)_j\right|} \leq \begin{cases} C\rho^{j-1}, & \text{if } I(f(\mathbb{T}), \lambda) > 0, \\ C\rho^{N-j}, & \text{if } I(f(\mathbb{T}), \lambda) < 0, \end{cases} \quad 1 \leq j \leq N, \ \text{for some } \rho > 1.$$

Non-Hermitian skin effect



$f(\mathbb{T})$ and the eigenvalues.



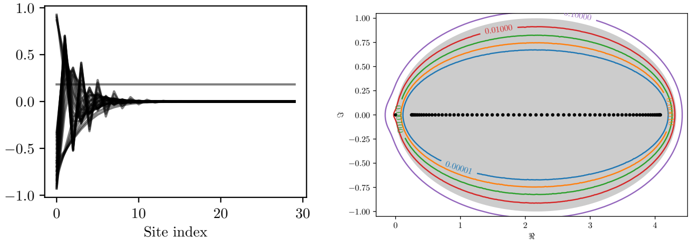
Eigenmodes.

Non-Hermitian skin effect

- One-dimensional case: $\mathcal{C}^\gamma$ tridiagonal, perturbed Toeplitz;
- $$C^\gamma = \begin{pmatrix} \alpha+\beta & \eta & & & \\ \beta & \alpha & \ddots & & \\ & \ddots & \ddots & & \\ & & & \alpha & \eta \\ & & & \beta & \alpha+\eta \end{pmatrix};$$
  $\alpha + \beta + \eta = 0$;
- $$\beta \neq \eta; |\gamma| \propto |\ln(\beta/\eta)|.$$
- Eigenvectors:
  $$\left| v_k^{(j)} \right| \lesssim e^{-|\gamma| \frac{j-1}{2}}.$$
- Eigenvalues: all real.

Non-Hermitian skin effect

- Eigenvector localisation and ϵ-pseudospectra of $\mathcal{C}^{\gamma}$:



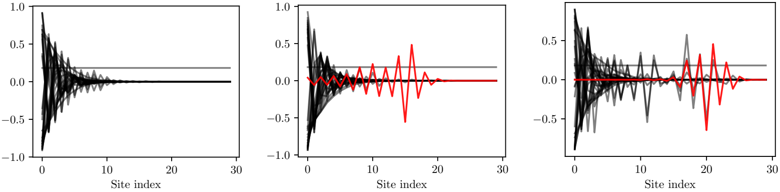
- Condensation of the eigenmodes at one edge; "Infinite" order exceptional point.
- Topological nature of the skin effect: localisation of the eigenmodes corresponding to eigenvalues $\in$ region where the symbol of the Toeplitz operator corresponding to the semi-infinite structure has negative winding.

Non-Hermitian skin effect

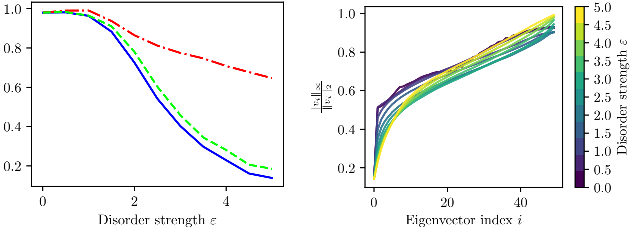
- Stability of the skin effect:
  - Topological protection of the associated (real) eigenfrequencies;
  - Competition between the non-Hermitian skin effect and the disorder-induced Anderson localisation;
  - As the strength of the disorder increases, more and more eigenmodes become localised in the bulk.

Non-Hermitian skin effect

- Single realisations with increasing disorder strengths:

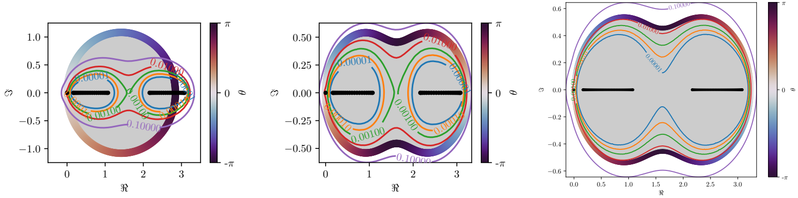


- Competition between the non-Hermitian skin effect and Anderson localisation:



Dimer systems

- Dimer systems $\Rightarrow$ Perturbed Block Toeplitz matrices.
- Fredholm index of the associated operator (= winding of the determinant of its symbol) takes value zero at some point on the unit circle.
- Winding of the two eigenvalues of the symbol: predicts accurately the exponential decay of the eigenmodes and is the limit of the pseudospectrum as $N \to \infty$.



Non-Hermitian Anderson model

- Tight-binding model:

$$H_{\rm tb} = \begin{pmatrix} e & e^{\gamma} & & \\ e^{-\gamma} & e & e^{\gamma} & \\ & \ddots & \ddots & \ddots \\ & & e^{-\gamma} & e \end{pmatrix}.$$

- Disorder: perturb only the diagonal entries.
- Skin effect model: correlated perturbations of all the entries; dense matrix model $\Leftarrow$ long-range coupling in three dimensions.

Space-time modulated systems of resonators

- Wave equation in a space-time modulated systems:

$$\left(\frac{\partial}{\partial t} \frac{1}{\kappa(x,t)} \frac{\partial}{\partial t} - \nabla \cdot \frac{1}{\rho(x)} \nabla \right) u(x,t) = 0, \quad x \in \mathbb{R}^d, t \in \mathbb{R}.$$

- Y: unit cell; $\mathcal{D} = \bigcup_{m \in \Lambda} D + m$; $\mathcal{D}_i = \bigcup_{m \in \Lambda} D_i + m$; $D_i, i = 1, \ldots, N$.
- Time-modulation of the resonators:

$$\kappa(x,t) = \begin{cases} \kappa, & x \in \mathbb{R}^d \setminus \overline{\mathcal{D}}, \\ \kappa_r \kappa_i(t), & x \in \mathcal{D}_i, \end{cases}, \qquad \kappa(x, t+T) = \kappa(x,t);$$

- Time-Brillouin zone: $\omega \in Y_t^* := \mathbb{C}/(\Omega\mathbb{Z})$; $\Omega = (2\pi)/T = O(\delta^{1/2})$.
- A quasifrequency is a subwavelength quasifrequency if the corresponding solution is essentially supported in the subwavelength frequency regime.

Space-time modulated systems of resonators

- Floquet transform in both x and t:

$$\begin{cases} \left(\frac{\partial}{\partial t} \frac{1}{\kappa(x,t)} \frac{\partial}{\partial t} - \nabla \cdot \frac{1}{\rho(x)} \nabla \right) u(x,t) = 0, \\ u(x,t) e^{-i\alpha\cdot x} \text{ is } \Lambda\text{-periodic in } x, \\ u(x,t) e^{-i\omega t} \text{ is } T\text{-periodic in } t. \end{cases}$$

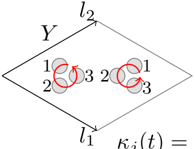
- Space-Brillouin zone: $\alpha \in Y^* := \mathbb{R}^d / \Lambda^*$; Time-Brillouin zone: $\omega \in Y_t^* := \mathbb{C}/(\Omega\mathbb{Z})$; $\Omega = (2\pi)/T$.
- As $\delta \to 0$, the quasifrequencies $\omega = \omega(\alpha) \in Y_t^*$ are, to leading order, given by the quasifrequencies of the system of ordinary differential equations:

$$\sum_{j=1}^{N} C_{ij}^{\alpha} \Phi_j = -\frac{d}{dt} \left(\frac{1}{\kappa_i} \frac{d\Phi_i}{dt} \right),$$

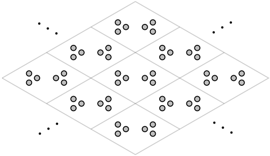
for $i = 1, \ldots, N$. $(\Phi_j(t) = e^{i\omega t} \sum_n \Phi_{j,n} e^{in\Omega t})$.

Pseudo-spin effect

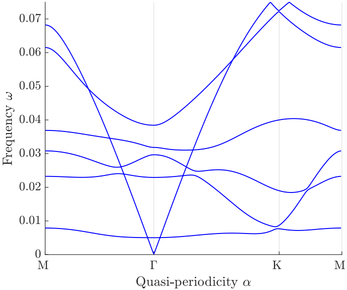
- Trimer honeycomb lattice with phase-shifted time-modulations inside the trimers:



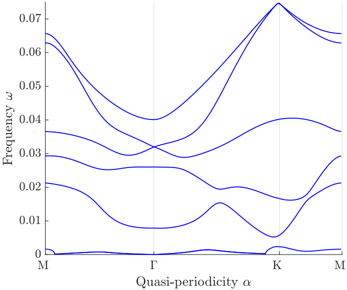
$$\kappa_j(t) = 1 + \epsilon \sin\left(\Omega t + \frac{2\pi j}{3}\right)$$



- Dirac cones at the origin of the Brillouin zone:



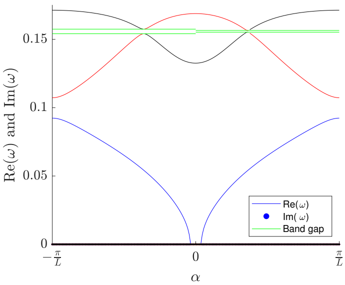
Unmodulated case



Modulated case

Non-reciprocal wave propagation and k-gaps

- Folding of the static band structure might create degenerate points;
- Degenerate points give rise to broken reciprocity;
- Non-reciprocal band gaps and k-gaps:



- Breaking reciprocity (time-reversal symmetry) $\Rightarrow$ non-symmetric bandgaps $\Rightarrow$ unidirectional excitation of the operating waves.
- Existence of k-gaps $\Rightarrow$ exponentially growing wave propagation.

Open questions

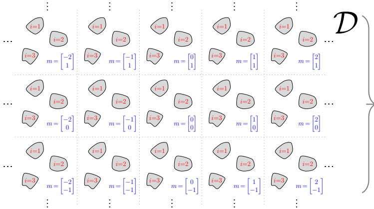
- Non-Hermitian time-varying systems of subwavelength resonators.
- Double-near-zero effective properties.
- Competition between the skin effect and Anderson localisation in three dimensions; non-Hermitian Thouless criterion.
- Space-time localisation.

Lecture IX: Convergence results for large systems of subwavelength resonators

Spectral convergence in large finite resonator arrays

- Relations between the finite structure and corresponding infinite one.
- Spectral convergence of defect modes:
  - Any defect mode eigenfrequency of the infinite structure has a sequence of eigenvalues of the truncated structures converging to it.
  - Long-range interactions $\Rightarrow$ radiation in the "spare" dimensions and coupling with the far field $\Rightarrow$ algebraic convergence;
  - No spare dimensions $\Rightarrow$ exponential convergence.
- Spectral convergence to the essential spectrum and band structure:
  - Subwavelength resonant frequencies of a system of coupled resonators in a truncated periodic lattice converge to the essential spectrum of corresponding infinite lattice.
  - Asymptotic distribution of the eigenvalues of the capacitance matrix, in the limit that its size becomes arbitrarily large.
  - Discrete density of states for the finite system converge in distribution to the continuous density of states of the infinite system.

Convergence of capacitance coefficients

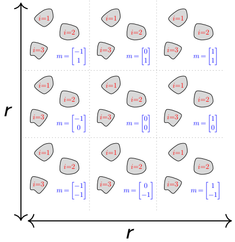


$\mathcal{D}$

$C^\alpha, \alpha \in Y^*$
quasiperiodic capacitance matrix

Inverse Floquet transform

$\mathfrak{C}$
real-space capacitance matrix



$\mathcal{D}_\mathrm{f}(r)$

$C_\mathrm{f}(r), r \in (0, \infty)$
finite capacitance matrix

$C_\mathrm{t}(r), r \in (0, \infty)$
truncated capacitance matrix

"Real-space" capacitance matrix

- C^α : "dual-space" representation of the infinite periodic system.
- Inverse Floquet transform $\Rightarrow$ "real-space" capacitance matrix at $m \in \Lambda$:

$$\widehat{C}_{ij}^m = \frac{1}{|Y^*|} \int_{Y^*} C_{ij}^\alpha e^{-\mathrm{i}\alpha\cdot m}\, \mathrm{d}\alpha, \quad 1 \le i,j \le N.$$

- $\mathfrak{C}$: infinite matrix that contains all the $\widehat{C}_{ij}^m$ coefficients, for all $1 \le i,j \le N$ and all $m \in \Lambda$:

$$\mathfrak{C} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \widehat{C}^0 & \widehat{C}^1 & \widehat{C}^2 & \widehat{C}^3 & \cdots \\ \cdots & \widehat{C}^{-1} & \widehat{C}^0 & \widehat{C}^1 & \widehat{C}^2 & \cdots \\ \cdots & \widehat{C}^{-2} & \widehat{C}^{-1} & \widehat{C}^0 & \widehat{C}^1 & \cdots \\ \cdots & \widehat{C}^{-3} & \widehat{C}^{-2} & \widehat{C}^{-1} & \widehat{C}^0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Spectral convergence of defect modes

- $$(C_{\mathrm{f}}^{mn})_{ij}(r) = \int_{\partial D_i^m} (\mathcal{S}_{\mathcal{D}_{\mathrm{f}}}^0)^{-1}[\chi_{\partial D_j^n}] \, \mathrm{d}\sigma,$$

  for $1 \leq i, j \leq N$ and $m, n \in I_r := \{m \in \Lambda \mid |m| < r\}$, the capacitance matrix for a finite lattice I_r.
- "Dual-space" representation of the quasiperiodic capacitance matrix for the infinite lattice:

  $$C_{ij}^{\alpha} = \int_{\partial D_i} (\mathcal{S}_D^{0,\alpha})^{-1}[\chi_{\partial D_j}] \, \mathrm{d}\sigma;$$
- "Real-space" capacitance coefficients at the lattice point m:

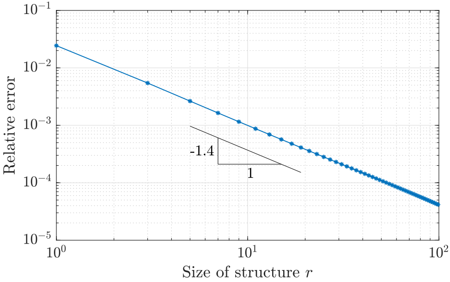
  $$\widehat{C}_{ij}^{m} = \frac{1}{|Y^*|} \int_{Y^*} C_{ij}^{\alpha} e^{-\mathrm{i}\alpha\cdot m} \, \mathrm{d}\alpha,$$

  for $1 \leq i, j \leq N$ and $m \in \Lambda$.
- Convergence of capacitance coefficients: For fixed $m, n \in \Lambda$, as $r \to \infty$,

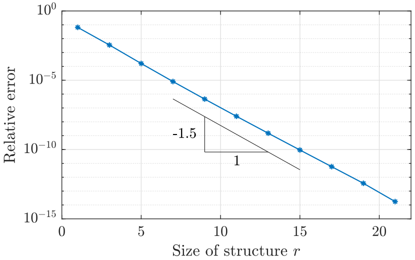
  $$\lim_{r\to\infty} C_{\mathrm{f}}^{mn}(r) = \widehat{C}^{m-n}.$$

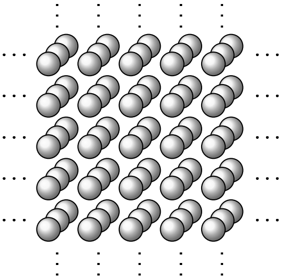
Convergence of capacitance coefficients

- $|(C_\mathrm{f})^0_{11} - \widehat{C}^0_{11}|$ for increasing size r of the finite structure: algebraic ($d_l < d$)/exponential ($d = d_l$) convergence.
- $d_l < d$: long range interactions in the "spare" dimensions.



Convergence of capacitance coefficients





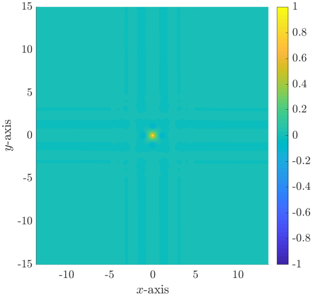
Convergence of capacitance coefficients

- C_t: Toeplitz matrix of an essentially bounded symbol;
- As $r \to \infty$, the matrices C_t and C_f are asymptotically equivalent:
  - $\lim_{r\to\infty} |C_\mathrm{f} - C_\mathrm{t}| = 0$;
  - $\|C_\mathrm{f}\|_2$ and $\|C_\mathrm{t}\|_2$ are uniformly bounded as $r \to \infty$.
- For an $n \times n$ matrix $M = (m_{ij})$, normalised Frobenius norm:

$$|M|^2 = \frac{1}{n} \sum_{i,j=1}^{n} |m_{ij}|^2.$$

- Asymptotically equivalent matrices have identical eigenvalue distributions as their sizes tend to infinity.

Spectral convergence of defect modes



Spectral convergence of defect modes

- Model defect modes through premultiplication by a defect matrix $\mathfrak{B}$. For each $m \in \Lambda$, $\mathcal{B}_m$: $N \times N$ diagonal matrix

$$\mathcal{B}_m = \begin{pmatrix} b_1^m & 0 & \cdots & 0 \\ 0 & b_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_N^m \end{pmatrix};$$

- Diagonal entries b_i^m: real-valued parameters.
- Compact defects: $b_i^m = 1$ for all but finitely many i and m.
- $\mathfrak{B}$: infinite block-diagonal matrix that contains $\mathcal{B}_m$ for all $m \in \Lambda$.
- Compact perturbation of the identity $\Rightarrow$ spectrum of the infinite structure given by the solutions to the spectral problem

$$\mathfrak{B}\mathfrak{C}\mathfrak{u} = \lambda \mathfrak{u};$$

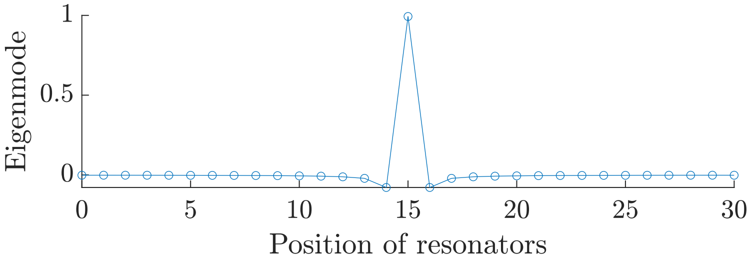
$(\lambda = \omega^2)$.

- Finite structure of size r: Let B_t be the block-diagonal matrix $(\mathcal{B}_m), m \in I_r$ and consider the spectral problem

$$B_\mathrm{t} C_\mathrm{f} u = \lambda u.$$

Spectral convergence of defect modes

- Example of a localised defect mode for a system of 31 resonators.
- The eigenvalues of the finite matrix $B_\mathrm{t} C_\mathrm{f}$ are computed, where C_f: capacitance matrix for a system of evenly spaces resonators and B_t: identity matrix but with the central entry $(B_\mathrm{t})^0_{11} = 2$.



Spectral convergence of defect modes

- Example of a defect structure:
  - Lattice with a single resonator $N = 1$ inside each unit cell; $x > -1$;
  - $$b_1^m = \begin{cases} 1, & m \neq 0, \\ 1 + x, & m = 0. \end{cases}$$
  - Eigenvalues of the (infinite-dimensional) generalised capacitance matrix $\mathfrak{BC}$: $\lambda(= \omega^2)$: an eigenvalue of $\mathfrak{BC}$ iff it is a root of
    $$\frac{x}{|Y^*|} \int_{Y^*} \frac{\lambda_1^\alpha}{\omega^2 - \lambda_1^\alpha} \, \mathrm{d}\alpha = 1;$$
  - λ_1^α: the single eigenvalue of the quasiperiodic capacitance matrix C^α of the unperturbed periodic structure,
    $$\lambda_1^\alpha := \sqrt{\frac{\delta_1 v_1^2}{|D_1|} (\mathcal{S}_{D_1}^{0,\alpha})^{-1} [\chi_{\partial D_1}]}.$$
  - $\exists$ solution $\lambda = \lambda_0 (= \omega_0^2)$ precisely in the case $x > 0 \Rightarrow$ defect induces an eigenvalue λ_0 in the pure point spectrum of $\mathfrak{BC}$, corresponding to an exponentially localised eigenmode.

Spectral convergence in large finite resonator arrays

- If the infinite structure has a localised mode, there will be an eigenvalue of the truncated structure arbitrarily close to the localised frequency.
- Assume that $\mathfrak{B}$: compact perturbation of the identity s.t. $\mathfrak{B}\mathfrak{C}$ has a localised eigenmode $\mathfrak{u}$ with corresponding eigenvalue $\lambda \Rightarrow \exists$ eigenvalue $\widetilde{\lambda} = \widetilde{\lambda}(r)$ of $B_{\mathrm{t}} C_{\mathrm{t}}$ satisfying

$$\lim_{r \to +\infty} \widetilde{\lambda}(r) = \lambda.$$

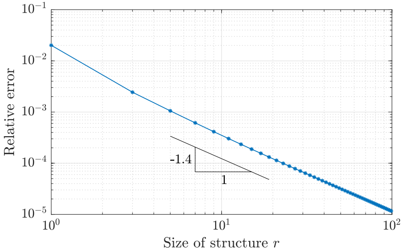
- Assume that $\mathfrak{B}$: compact perturbation of the identity s.t. $\mathfrak{B}\mathfrak{C}$ has a localised eigenmode $\mathfrak{u}$ with corresponding eigenvalue $\lambda \Rightarrow \exists$ eigenvalue $\widehat{\lambda} = \widehat{\lambda}(r)$ of $B_{\mathrm{t}} C_{\mathrm{f}}$ satisfying

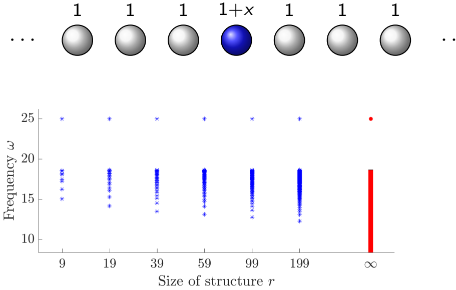
$$\lim_{r \to +\infty} \widehat{\lambda}(r) = \lambda.$$

Spectral convergence in large finite resonator arrays

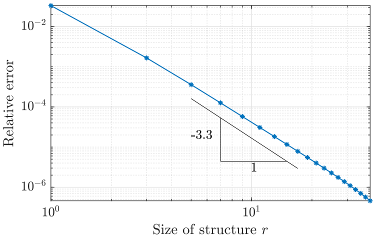
- Convergence of the difference between the defect frequency computed for a finite structure and for the corresponding infinite structure, computed analytically.
- Error of the frequency of the defect mode: inheriting the convergence rate of the capacitance coefficients.
- When $d_l = 1$ or $d_l = 2$, there are long-range interactions through coupling with the far field, leading to algebraic convergence.
- In $d_l = 3$, there are no "spare" dimensions and the convergence is exponential.

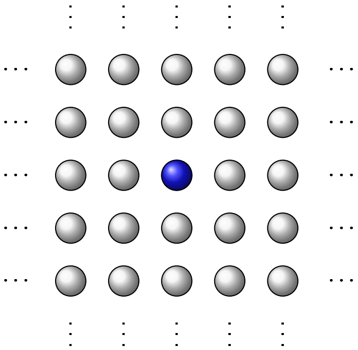
Spectral convergence in large finite resonator arrays



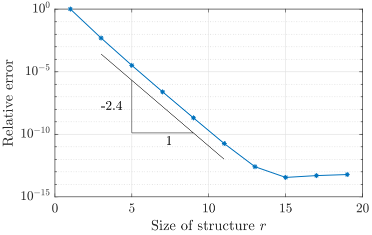


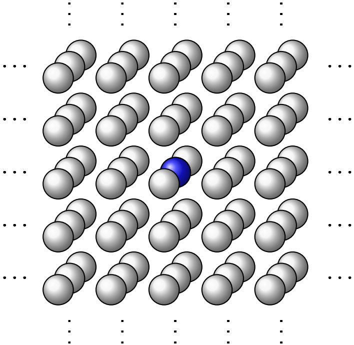
Spectral convergence in large finite resonator arrays





Spectral convergence in large finite resonator arrays

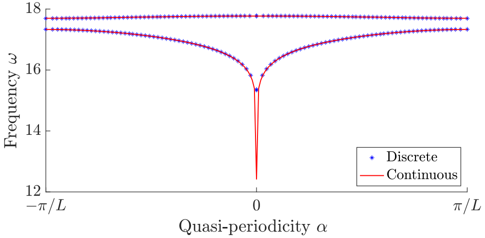


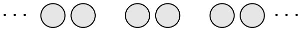


Spectral convergence in large finite resonator arrays

- Pointwise convergence to the essential spectrum: Any eigenvalue/eigenvector of $\mathcal{C}^\alpha$ can be approximated by eigenvalues/eigenvectors of $\mathcal{C}_\mathrm{f}$; Converse not true: edge effect $\Rightarrow$ greatest effect on eigenmodes within the first radiation continuum.
- Convergence in distribution of the discrete density of states for the finite M-system of N periodically repeated resonators to the (continuous) density of states of the infinite system:

$$D_\mathrm{f}(\omega) := \frac{1}{MN}\sum_{j=1}^{MN}\delta\Big(\omega-\omega_j^{(M)}\Big) \to D(\omega) := \frac{1}{N}\sum_{k=1}^{N}\int_{Y^*}\delta\Big(\omega-\hat{\omega}_k(\alpha)\Big)\,d\alpha.$$





Spectral convergence in large finite resonator arrays

- Weak convergence of $\mathcal{C}_\mathrm{f}$ ($M \times M$-block matrix with blocks of size N) to corresponding (translationally invariant) Toeplitz matrix $\mathcal{C}_\mathrm{t}$ of the infinite structure.
- $\mathcal{C}^m$: inverse Floquet transform of $\mathcal{C}^\alpha$ (real-space capacitance matrix);
- $\mathfrak{C}$: (block) Laurent operator corresponding to the symbol $\mathcal{C}^\alpha$:

$$\mathfrak{C} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \mathcal{C}^0 & \mathcal{C}^1 & \mathcal{C}^2 & \mathcal{C}^3 & \cdots \\ \cdots & \mathcal{C}^{-1} & \mathcal{C}^0 & \mathcal{C}^1 & \mathcal{C}^2 & \cdots \\ \cdots & \mathcal{C}^{-2} & \mathcal{C}^{-1} & \mathcal{C}^0 & \mathcal{C}^1 & \cdots \\ \cdots & \mathcal{C}^{-3} & \mathcal{C}^{-2} & \mathcal{C}^{-1} & \mathcal{C}^0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

- $\mathcal{C}_t$: Toeplitz matrix with symbol $\mathcal{C}^\alpha$:

$$\mathcal{C}_\mathrm{t} = \begin{pmatrix} \mathcal{C}^0 & \mathcal{C}^1 & \cdots & \mathcal{C}^M \\ \mathcal{C}^{-1} & \mathcal{C}^0 & \cdots & \mathcal{C}^{M-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{C}^{-M} & \mathcal{C}^{1-M} & \cdots & \mathcal{C}^0 \end{pmatrix}.$$

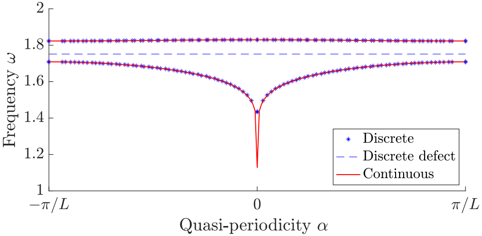
- $\mathcal{C}_\mathrm{f}, \mathcal{C}_\mathrm{t}$ asymptotically equivalent: $\frac{1}{\sqrt{M}}\|\mathcal{C}_\mathrm{f} - \mathcal{C}_\mathrm{t}\|_F \to 0$; $\|\mathcal{C}_\mathrm{f}\|_2, \|\mathcal{C}_\mathrm{t}\|_2$ uniformly bounded.
- $\mathcal{C}_\mathrm{f}, \mathcal{C}_\mathrm{t}$: identical eigenvalue distributions as their sizes $\to \infty$.

Spectral convergence in large finite resonator arrays

- Truncated Floquet transform: $(\omega_j, u_j), (u_j)_m$: vector of length N associated to cell $m \in \Lambda$;

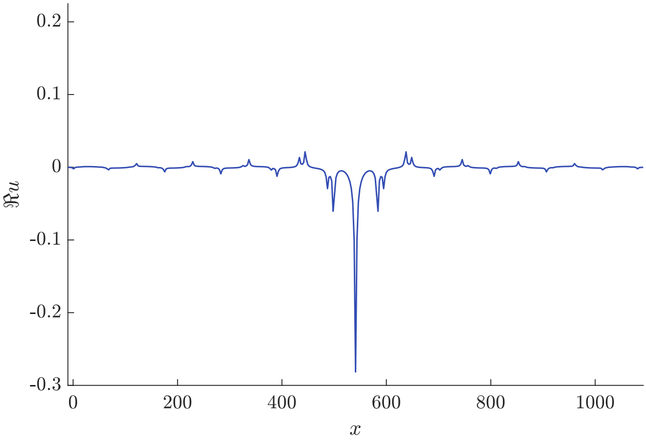
$$(\widehat{u_j})_\alpha = \sum_{m \in \text{finite lattice}} (u_j)_m e^{i\alpha \cdot m}; \quad \alpha_j = \operatorname*{arg\,max}_{\alpha \in Y^*} \|(\widehat{u_j})_\alpha\|_2.$$

- Principle applicable to structures that are not translationally invariant:

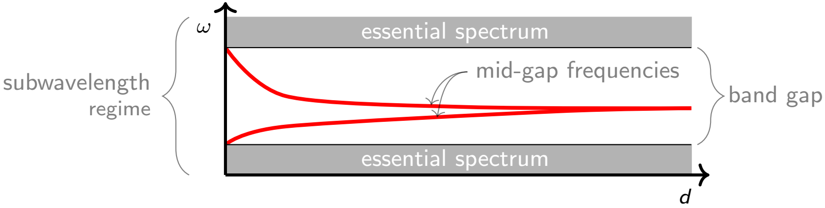


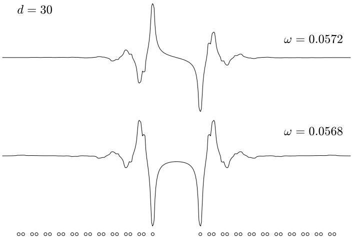
- Defect modes in infinite systems of resonators have corresponding modes in finite systems which converge as the size of the system increases.

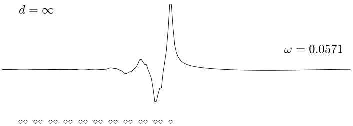
Spectral convergence in large finite resonator arrays



Spectral convergence in large finite resonator arrays





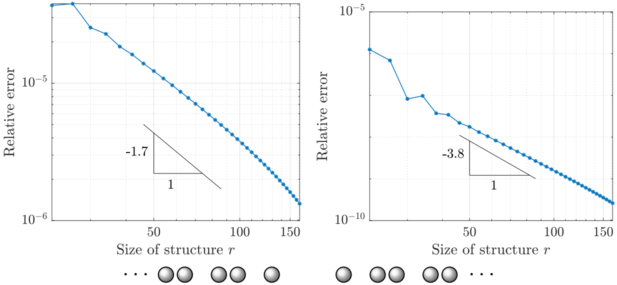


Spectral convergence in large finite resonator arrays

- Rate of convergence in terms of the length $r = O(M)$ of the truncated structure:

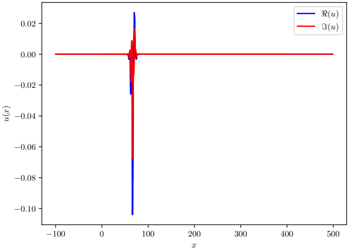
$$d_l = d \Rightarrow \text{exponential}; \quad d_l < d \Rightarrow \text{algebraic}.$$

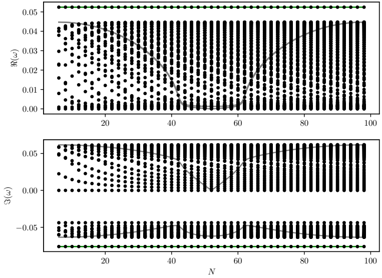
- Algebraic convergence $\Leftarrow$ long-range interactions due to coupling with the far-field.
- Convergence of the frequency of the defect modes in a dislocated chain.
- $O(r^{-1.7})$ for the even mode and $O(r^{-3.8})$ for the odd mode:



Convergence results for non-Hermitian large systems

- Parity-time symmetric systems; Edge mode computed for a finite but large ($N = 100$) array of resonators having a material parameter defect;
- One dimension: comparison with the explicit formula for the edge mode frequency.

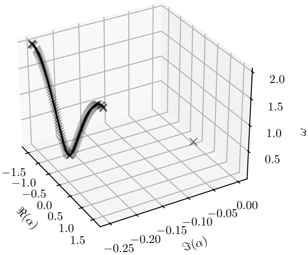




Convergence results for non-Hermitian large systems

- Non-Hermitian skin effect.
- Spectrum of the limiting operator: Non-Bloch eigenmodes $\Rightarrow$ generalised (complex) Brillouin zone

  $\mathcal{Y}^* := \{(\alpha, \beta(\alpha)) \in Y^* \times \mathbb{R} : \lambda^{\alpha+i\beta(\alpha)} \in \mathbb{R}^+\}$; $\lambda^{\alpha+i\beta(\alpha)}$ eigenvalue of $\mathcal{C}^{\gamma,\alpha+i\beta(\alpha)}$.
- Convergence to the complex band structure:



- Systems with complex material parameters can be reduced to Hermitian systems away from their exceptional points.
- Non-Hermitian systems with imaginary gauge potentials / Non-Hermitian systems with complex material parameters: fundamentally distinct.

Convergence results for non-Hermitian large systems

- Subwavelength eigenfrequencies:

$$\psi'' + VC\psi = 0;$$

V: diagonal matrix encoding the (complex) material parameters; C: (Hermitian) capacitance matrix.

- Assume that VC: diagonalisable and invertible, i.e., we are away from exceptional points.
- Change of basis $\Rightarrow VC = D = \text{diag}(\lambda_1, \lambda_2)$.
- Transformation:

$$G : \mathbb{R} \to \mathbb{C}^2$$

$$t \mapsto (\lambda_1^{-\frac{1}{2}} t, \lambda_2^{-\frac{1}{2}} t);$$

- $\phi(t) = \psi \circ G(t)$ satisfies

$$\phi''(t) = D \cdot^{-1} (\psi'' \circ G(t)).$$

- $\Rightarrow \phi(t)$ satisfies the Hermitian ODE

$$\phi'' + \phi = 0$$

as

$$D(\phi''(t) + \phi(t)) = DD(^{-1}\psi'' \circ G(t)) + D(\psi \circ G(t)) = (\psi'' + D\psi) \circ G(t) = 0.$$

Open questions

- Truncated Floquet transform.
- Localised eigenmodes in finite chains of subwavelength resonators.
- Approximations of Fano-type transmition and reflection behaviors by finite structures.

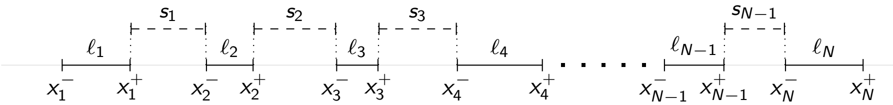
Lecture X: When subwavelength physics meets condensed matter theory and concluding remarks

One-dimensional subwavelength physics

- One-dimensional subwavelength resonator systems
- A chain D of N resonators, with lengths $(\ell_i)_{1\leq i\leq N}$ and spacings $(s_i)_{1\leq i\leq N-1}$.
- Model problem:

$$\frac{\omega^2}{\kappa(x)}u(x) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\rho(x)}\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) = 0, \qquad x \in \mathbb{R}.$$

- $$\kappa(x) = \begin{cases} \kappa_r, & x \in D, \\ \kappa, & x \in \mathbb{R}\setminus D, \end{cases} \qquad \rho(x) = \begin{cases} \rho_r, & x \in D, \\ \rho, & x \in \mathbb{R}\setminus D. \end{cases}$$



One-dimensional subwavelength physics

- Tridiagonal capacitance matrix:

$$C = \begin{pmatrix} \frac{1}{s_1} & -\frac{1}{s_1} & & & & \\ -\frac{1}{s_1} & \frac{1}{s_1} + \frac{1}{s_2} & -\frac{1}{s_2} & & & \\ & -\frac{1}{s_2} & \frac{1}{s_2} + \frac{1}{s_3} & -\frac{1}{s_3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{s_{N-2}} & \frac{1}{s_{N-2}} + \frac{1}{s_{N-1}} & -\frac{1}{s_{N-1}} \\ & & & & -\frac{1}{s_{N-1}} & \frac{1}{s_{N-1}} \end{pmatrix}.$$

- C: symmetric, semi-definite;
- $\mathrm{Ker}\, C = \mathrm{span}\big((1)_{1\leq i\leq N}\big)$.

One-dimensional subwavelength physics

- N subwavelength resonant frequencies ω_i:

$$\omega_i = v_b\sqrt{\delta\lambda_i} + \mathcal{O}(\delta),$$

$(\lambda_i)_{1\leq i\leq N}$: eigenvalues of the generalised eigenvalue problem

$$C\boldsymbol{a}_i = \lambda_i V\boldsymbol{a}_i \qquad 1\leq i\leq N;$$

$V := \mathrm{diag}\left((\ell_i)_{1\leq i\leq N}\right)$.

- $(\boldsymbol{a}_i)_i$ orthornomal basis with respect to the scalar product of V:

$$\boldsymbol{a}_i^\top V\boldsymbol{a}_j = \delta_{ij}, 1\leq i,j\leq N.$$

- $\boldsymbol{a}_1 = (1/\sqrt{\sum_{i=1}^N l_i})\,\mathbf{1}$.
- $u_i(x)$: subwavelength eigenmode corresponding to ω_i; $\boldsymbol{a}_i$: corresponding eigenvector of $\mathcal{C}$:

$$u_i(x) = \sum_{j=1}^N \boldsymbol{a}_i^{(j)} V_j(x) + \mathcal{O}(\delta)$$

- $\boldsymbol{a}_i^{(j)}$: j-th entry of the eigenvector $\boldsymbol{a}_i$; $V_j(x)$: piecewise linear, supported in $(x_{j-1}^{\mathrm{R}}, x_{j+1}^{\mathrm{L}})$ and $V_j(x) = 1$ for $x\in(x_j^{\mathrm{L}}, x_j^{\mathrm{R}})$.

One-dimensional subwavelength physics

- The N eigenvalues of the capacitance matrix C are simple:

$$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_N.$$

- The scattering problem admits exactly $2N$ resonant frequencies:
  - the zero frequency $\omega_0(\delta) = 0$ for any $\delta > 0$,
  - a purely imaginary frequency $\omega_1(\delta)$, which is an analytic function of δ whose leading asymptotic expansion reads:

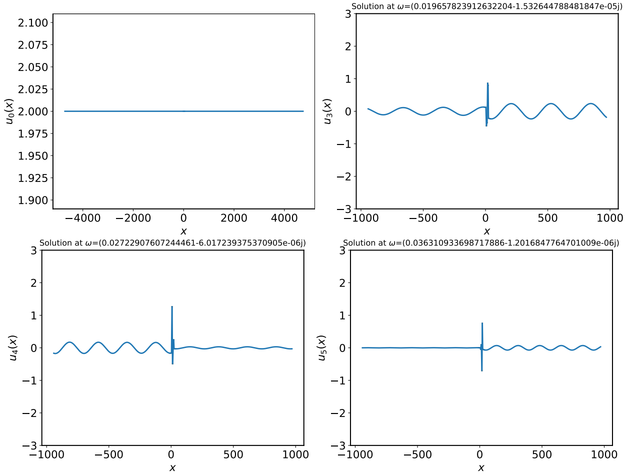
$$\omega_1(\delta) = -2\mathrm{i}\delta \frac{v_b^2}{v \sum_{j=1}^N \ell_j} + O(\delta^2);$$

  - the remaining $2N-2$ resonant frequencies are analytic functions of $\delta^{\frac{1}{2}}$ and their leading-order asymptotic expansion read

$$\omega_i^{\pm}(\delta) = \pm v_b \lambda_i^{\frac{1}{2}} \delta^{\frac{1}{2}} - \mathrm{i}\delta \frac{v_b^2}{2v} \boldsymbol{a}_i^\top B \boldsymbol{a}_i + O(\delta^{\frac{3}{2}}) \text{ for } 2 \leq i \leq N;$$

$B := \mathrm{diag}(1, 0, \cdots, 0, 1).$

One-dimensional subwavelength physics

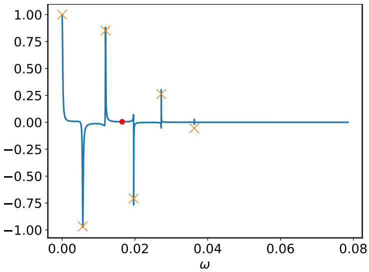


https://people.cs.kuleuven.be/~florian.feppon/research/06_subwavelength_resonances.html

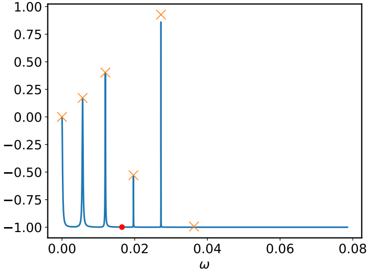
One-dimensional subwavelength physics

- u_{in}: an incident wave propagating from left to right.
- Transmission and reflection coefficients:

$$T(\omega,\delta) := \frac{u(x_N^+)}{u_{\text{in}}(x_N^+)}, \qquad R(\omega,\delta) := \frac{u(x_1^-) - u_{\text{in}}(x_1^-)}{u_{\text{in}}(x_1^-)}.$$



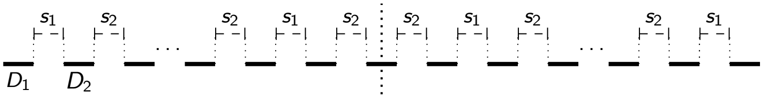
Transmission coefficient $\Re(T(\omega,\delta))$.



Reflection coefficient $\Re(R(\omega,\delta))$.

One-dimensional subwavelength physics

- Existence of a spectral gap for defectless finite dimer structures;
- Direct relationship between eigenvalues being within the spectral gap and the localisation of their associated eigenmode.
- Existence and uniqueness of an eigenvalue in the gap in the defect structure, proving the existence of a unique localised interface mode.
- Chebyshev polynomials: characterise quantitatively the localised interface modes in systems of finitely many resonators.
- Dimer structure with a geometric defect:



One-dimensional subwavelength physics

- Tridiagonal block structure:

$$
\mathcal{C} = \begin{pmatrix}
\tilde{\alpha} & \beta_1 & & & & & & & & & & & \\
\beta_1 & \alpha & \beta_2 & & & & & & & & & & \\
 & \beta_2 & \alpha & \beta_1 & & & & & & & & & \\
 & & \ddots & \ddots & \ddots & & & & & & & & \\
 & & & \beta_2 & \alpha & \beta_1 & & & & & & & \\
 & & & & \beta_1 & \alpha & \beta_2 & & & & & & \\
 & & & & & \beta_2 & \eta & \beta_2 & & & & & \\
 & & & & & & \beta_2 & \alpha & \beta_1 & & & & \\
 & & & & & & & \beta_1 & \alpha & \beta_2 & & & \\
 & & & & & & & & \ddots & \ddots & \ddots & & \\
 & & & & & & & & & \beta_1 & \alpha & \beta_2 & \\
 & & & & & & & & & & \beta_2 & \alpha & \beta_1 \\
 & & & & & & & & & & & \beta_1 & \tilde{\alpha}
\end{pmatrix}
$$

$$
\beta_1 = -s_1^{-1}, \quad \beta_2 = -s_2^{-1}, \quad \alpha = s_1^{-1} + s_2^{-1}, \quad \eta = 2s_2^{-1}, \quad \tilde{\alpha} = s_1^{-1}.
$$

One-dimensional subwavelength physics

- Eigenvalues and eigenvectors of tridiagonal 2-Toeplitz matrices with perturbations on the corners:

$$A_{2k+1}^{(a,b)}(\alpha,\beta_1,\beta_2) := \begin{pmatrix} \alpha+a & \beta_1 & 0 & 0 & \dots & 0 & 0 \\ \beta_1 & \alpha & \beta_2 & 0 & \dots & 0 & 0 \\ 0 & \beta_2 & \alpha & \beta_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \alpha & \beta_2 \\ 0 & 0 & 0 & 0 & \dots & \beta_2 & \alpha+b \end{pmatrix} \in \mathbb{R}^{(2k+1)\times(2k+1)}$$

$$A_{2k}^{(a,b)}(\alpha,\beta_1,\beta_2) := \begin{pmatrix} \alpha+a & \beta_1 & 0 & 0 & \dots & 0 & 0 \\ \beta_1 & \alpha & \beta_2 & 0 & \dots & 0 & 0 \\ 0 & \beta_2 & \alpha & \beta_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \alpha & \beta_1 \\ 0 & 0 & 0 & 0 & \dots & \beta_1 & \alpha+b \end{pmatrix} \in \mathbb{R}^{2k\times 2k}.$$

One-dimensional subwavelength physics

- U_k: Chebyshev polynomial of the second kind;
- $$P_k^*(x) := (\beta_1\beta_2)^k \, U_k\left(\frac{(x-\alpha)^2 - \beta_1^2 - \beta_2^2}{2\beta_1\beta_2}\right),$$
- $$y(z) := \frac{z^2 - \beta_1^2 - \beta_2^2}{2\beta_1\beta_2}.$$
- Characteristic polynomials of $A_{2k+1}^{(a,b)}$ and $A_{2k}^{(a,b)}$:

$$\chi_{A_{2k+1}^{(a,b)}}(x) = (x - \alpha - a - b)\, P_k^*(x) + \left(ab\,(x-\alpha) - a\beta_1^2 - b\beta_2^2\right) P_{k-1}^*(x);$$

$$\chi_{A_{2k}^{(a,b)}}(x) = P_k^*(x) + \left((a+b)(\alpha - x) + ab + \beta_2^2\right) P_{k-1}^*(x) + ab\beta_1^2 P_{k-2}^*(x).$$

One-dimensional subwavelength physics

- Two families of polynomials $\widehat{p}_{k+1}^{(\xi_p,\xi_q)}(x)$ and $\widehat{q}_{k+1}^{(\xi_p,\xi_q)}(x)$: solutions to the recursion relations

$$\widehat{p}_0^{(\xi_p,\xi_q)}(\mu) = \xi_p, \quad \widehat{p}_1^{(\xi_p,\xi_q)}(\mu) = 2\mu\xi_p + \frac{\xi_p - \xi_q}{\beta},$$
$$\widehat{p}_{k+1}^{(\xi_p,\xi_q)}(\mu) = 2\mu\widehat{p}_k^{(\xi_p,\xi_q)}(\mu) - \widehat{p}_{k-1}^{(\xi_p,\xi_q)}(\mu),$$
$$\widehat{q}_0^{(\xi_p,\xi_q)}(\mu) = \xi_q, \quad \widehat{q}_1^{(\xi_p,\xi_q)}(\mu) = (2\mu + \beta)\xi_p + \frac{\xi_p - \xi_q}{\beta},$$
$$\widehat{q}_{k+1}^{(\xi_p,\xi_q)}(\mu) = 2\mu\widehat{q}_k^{(\xi_p,\xi_q)}(\mu) - \widehat{q}_{k-1}^{(\xi_p,\xi_q)}(\mu),$$

- $\beta = \beta_2/\beta_1$.

One-dimensional subwavelength physics

- λ: eigenvalue of $A_{2k+1}^{(a,b)}(\alpha, \beta_1, \beta_2)$. Corresponding eigenvector:

$$\boldsymbol{x} = \Big(\hat{q}_0^{(\xi_p,\xi_q)}(\mu), -\frac{1}{\beta_1}(\alpha-\lambda)\,\hat{p}_0^{(\xi_p,\xi_q)}(\mu), \hat{q}_1^{(\xi_p,\xi_q)}(\mu), -\frac{1}{\beta_1}(\alpha-\lambda)\,\hat{p}_1^{(\xi_p,\xi_q)}(\mu), \ldots, -\frac{1}{\beta_1}(\alpha-\lambda)\,\hat{p}_{k-1}^{(\xi_p,\xi_q)}(\mu), \hat{q}_k^{(\xi_p,\xi_q)}(\mu) \Big).$$

- λ: eigenvalue of $A_{2k}^{(a,b)}(\alpha, \beta_1, \beta_2)$. Corresponding eigenvector:

$$\boldsymbol{x} = \Big(\hat{q}_0^{(\xi_p,\xi_q)}(\mu), -\frac{1}{\beta_1}(\alpha-\lambda)\,\hat{p}_0^{(\xi_p,\xi_q)}(\mu), \hat{q}_1^{(\xi_p,\xi_q)}(\mu), -\frac{1}{\beta_1}(\alpha-\lambda)\,\hat{p}_1^{(\xi_p,\xi_q)}(\mu), \ldots, -\frac{1}{\beta_1}(\alpha-\lambda)\,\hat{p}_{k-1}^{(\xi_p,\xi_q)}(\mu) \Big).$$

- In both cases, $\xi_q = (\alpha - \lambda), \xi_p = (\alpha + a - \lambda)$.

One-dimensional subwavelength physics

- Structure of the eigenvectors for the capacitance matrix $\mathcal{C}$:
  Let $(\lambda, \boldsymbol{v})$ be an eigenpair of $\mathcal{C}$ and let $\mu := y(\lambda)$. Then $\boldsymbol{v}$:

$$\boldsymbol{v} = (\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots, \boldsymbol{x}^{(2m)}, \boldsymbol{x}^{(2m+1)}, (-1)^{\sigma}\boldsymbol{x}^{(2m)}, \dots, (-1)^{\sigma}\boldsymbol{x}^{(2)}, (-1)^{\sigma}\boldsymbol{x}^{(1)})^{\top};$$

- $\boldsymbol{x} \in \mathbb{R}^{2m+1}$ with $\xi_q = (\alpha - \lambda), \xi_p = (\alpha + a - \lambda)$; $\sigma \in \{0, 1\}$ except for $\boldsymbol{x} \in \mathrm{span}\{\mathbf{1}\}$ where $\sigma = 1$.

One-dimensional subwavelength physics

- Asymptotic spectral bulk Σ and asymptotic spectral gap Γ: spectral bulk and spectral gap of the associated infinite periodic system, respectively.
- Consider a system of repeated dimers (without defect) with $N = 2m$ resonators. C_N: associated capacitance matrix. Then

$$\Sigma = \overline{\lim_{N \to \infty} \sigma(C_N)} = \left[0, \frac{2}{s_2}\right] \cup \left[\frac{2}{s_1}, \frac{2}{s_1} + \frac{2}{s_2}\right];$$

- lim: Hausdorff limit.
- $\Rightarrow$ asymptotic spectral gap:

$$\Gamma = \left(\frac{2}{s_2}, \frac{2}{s_1}\right) \subset \mathbb{R}.$$

One-dimensional subwavelength physics

- $v(x)$: eigenmode. v: localised interface mode at x_0, if both $|v(x-x_0)|$ for $x_0 < x \in D$ and $|v(x_0 - x)|$ for $x_0 > x \in D$ decay exponentially as a function of $x \in D$.
- $\mathcal{C} \in \mathbb{R}^{4m+1\times 4m+1}$: capacitance matrix of the defect structure; (λ, v): an eigenpair of $\mathcal{C}$. Then, there exists $|r| \geq 1$ independent of m and $A, B, \tilde{A}, \tilde{B} \in \mathbb{R}$ dependent on m s.t.

  if $y(\lambda)^2 > 1$:

  $$v^{(|2m-2j|)} = Ar^j + Br^{-j},$$
  $$v^{(|2m-2j-1|)} = \tilde{A}r^j + \tilde{B}r^{-j};$$

  with $A = \mathcal{O}(\frac{1}{r^m})$ and $B = \mathcal{O}(r^{m-1})$ as $m \to \infty$ for $c_1, c_2 \in \mathbb{R}$ independent of m. Same asymptotics hold for $\tilde{A}$ and $\tilde{B}$;

  if $y(\lambda)^2 < 1$:

  $$v^{(|2m-2j|)} = A\cos(j\theta) + B\sin(j\theta),$$
  $$v^{(|2m-2j-1|)} = \tilde{A}\cos(j\theta) + \tilde{B}\sin(j\theta),$$

  with $r = e^{\mathrm{i}\theta}$ and $A, B, \tilde{A}, \tilde{B}$ bounded as $m \to \infty$;

One-dimensional subwavelength physics

if $y(\lambda)^2 = 1$: $r = \pm 1$ and

$$v^{(|2m-2j|)} = A r_1^j + B r_1^j \cdot j,$$
$$v^{(|2m-2j-1|)} = \tilde{A} r_1^j + \tilde{B} r_1^j \cdot j,$$

with $A = \frac{r^{1-m}(c_1 mr - c_1 r - c_2 m)}{mr^2 - m - r^2}$ and $B = \frac{r^m(c_2 r - c_1)}{mr^2 - m - r^2}$ as $m \to \infty$ for $c_1, c_2 \in \mathbb{R}$ independent of m. Same asymptotics hold for $\tilde{A}$ and $\tilde{B}$.

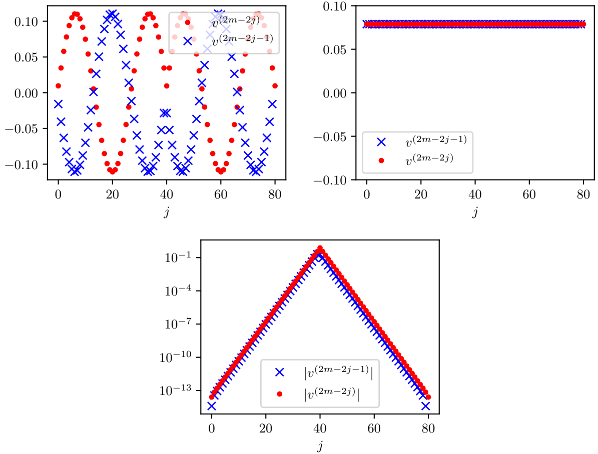
- Eigenvector in the case when $y(\lambda)^2 > 1$: exponentially localised in the interface, as we can rescale the eigenvector to make

$$v^{(|2m-2j|)} = B r^{-j} + A r^j,$$
$$v^{(|2m-2j-1|)} = \tilde{B} r^{-j} + \tilde{A} r^j,$$

where $\mathcal{O}(B) = \mathcal{O}(\tilde{B}) = \mathcal{O}(1)$ and $\mathcal{O}(A r^j) = \mathcal{O}(\tilde{A} r^j) = o(\frac{1}{r^{m-1}}), j = 1, \cdots, 2m$.

One-dimensional subwavelength physics

- Eigenvector behaviour based on the eigenvalue location:



One-dimensional subwavelength physics

- Perturbed structure of dimers. For N large enough there exists at least one localised interface eigenvector of $\mathcal{C}$ with eigenvalue $\lambda_i^{(N)}$ in the spectral gap Γ.
- Monotonicity of Chebyshev polynomials of the second kind: Let $k \in \mathbb{N}$, then

$$\frac{U_{k-1}(x)}{U_k(x)}$$

is strictly decreasing for $x \in (-\infty, -1) \cup (1, +\infty)$ for any $k \in \mathbb{N}$.
- $\Rightarrow$ There exists at most one eigenvalue of $\mathcal{C}$ lying in the asymptotic spectral gap $\Gamma = (2/s_2, 2/s_1)$. In particular, for m large enough, there exists exactly one eigenvalue in Γ.

One-dimensional subwavelength physics

- Convergence: Consider a perturbed structure of dimers. For N large enough there exists a unique interface mode with eigenfrequency $\omega_{\mathrm{i}}^{(N)}$ in the band gap. The associated eigenfrequency $\omega_{\mathrm{i}}^{(N)}$ converges to

$$\omega_{\mathrm{i}} = v_b \sqrt{\delta \frac{1}{2}\left(-\sqrt{\frac{9}{s_1^2} - \frac{14}{s_1 s_2} + \frac{9}{s_2^2}} + \frac{3}{s_1} + \frac{3}{s_2}\right)}$$

  exponentially as $N \to \infty$.

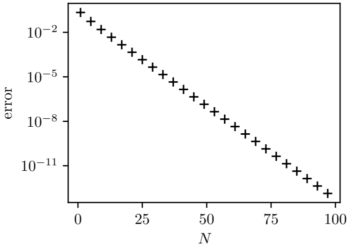
- In particular, for N big enough,

$$|\omega_{\mathrm{i}} - \omega_{\mathrm{i}}^{(N)}| < Ae^{-BN},$$

  for some $A, B > 0$ independent of N.

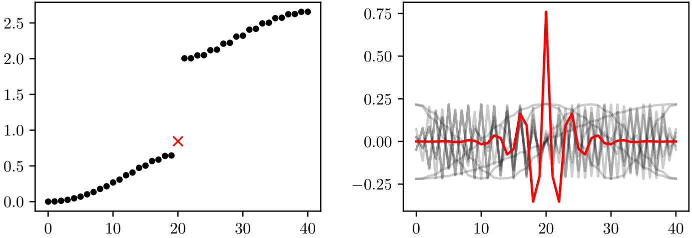
One-dimensional subwavelength physics

- Convergence of the interface mode eigenfrequency as the structure size increases:



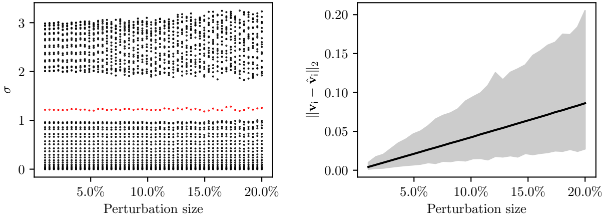
One-dimensional subwavelength physics

- Asymptotic spectral gap and interface mode:



One-dimensional subwavelength physics

- Stability of the interface mode:



Concluding remarks

- Mathematical foundations of subwavelength physics:
  - Localisation and topological properties of Hermitian, non-Hermitian and time-modulated systems of subwavelength resonators.
  - Dirac, exceptional point, fold degeneracies.
- Unified capacitance matrix framework for studying systems with long range interactions.
- Classification of non-hermitian problems into reciprocal and non-reciprocal ones.
- Non-reciprocity can be achieved by time-modulations.
- Many demonstrated quantum phenomena are not particular to quantum systems.
- First principle derivations for systems of subwavelength resonators.
- Subwavelength physics meets condensed matter theory in one dimension.
- Long-range interactions play key role in higher dimensions.