Many Body Problems in Quantum Optics

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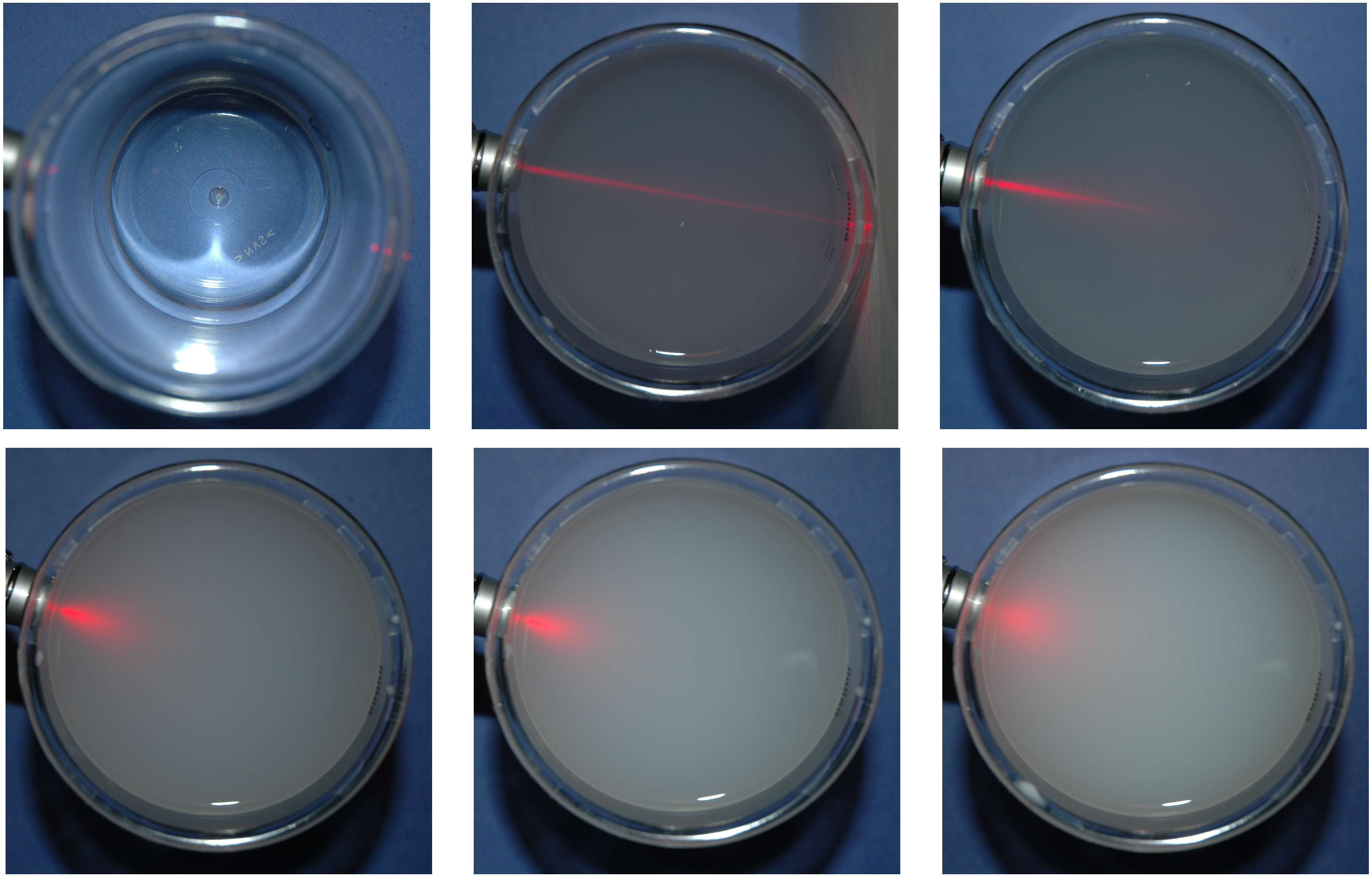
Disclaimer

Classical Optics

Light scattering

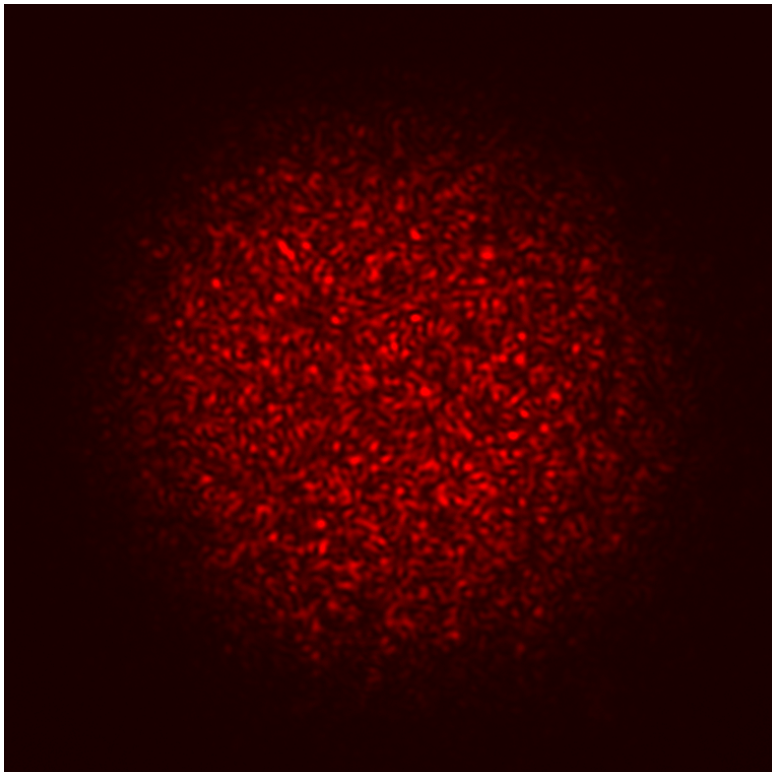


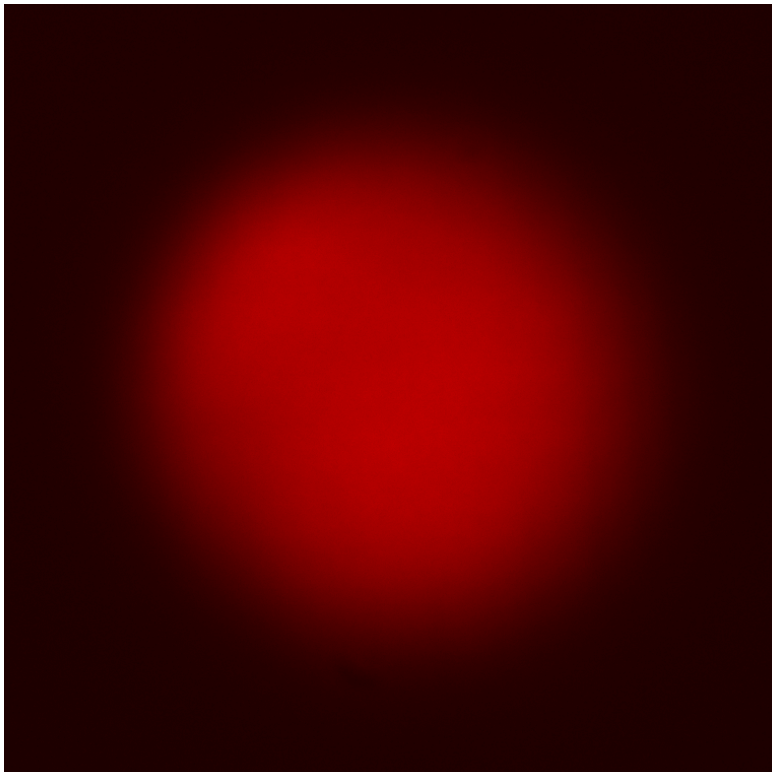
The particles undergo Brownian motion. The image can be understood as a time average over the paths of the particles.

Since the positions of the particles are unknown, it will prove to be advantageous to regard them as random.

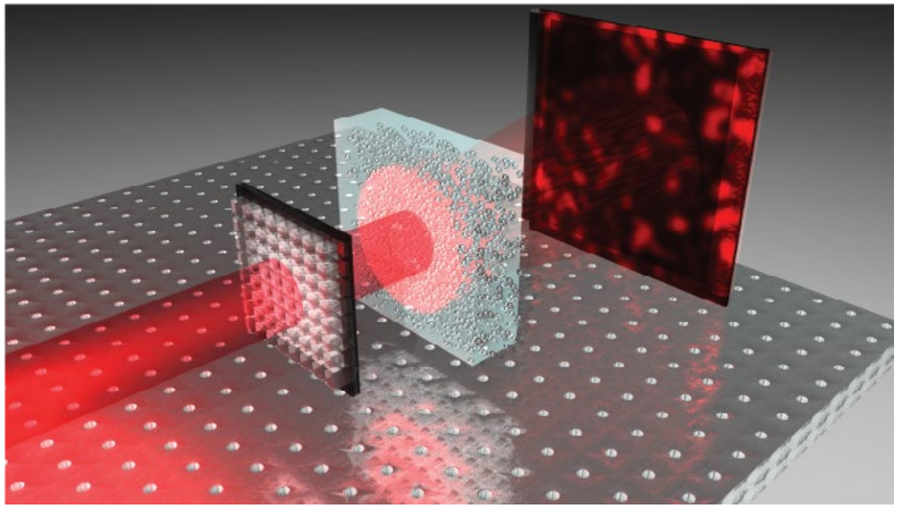
Suppose we have a fast camera and can freeze the motion of the particles; we can think of this as one realization of a random process.

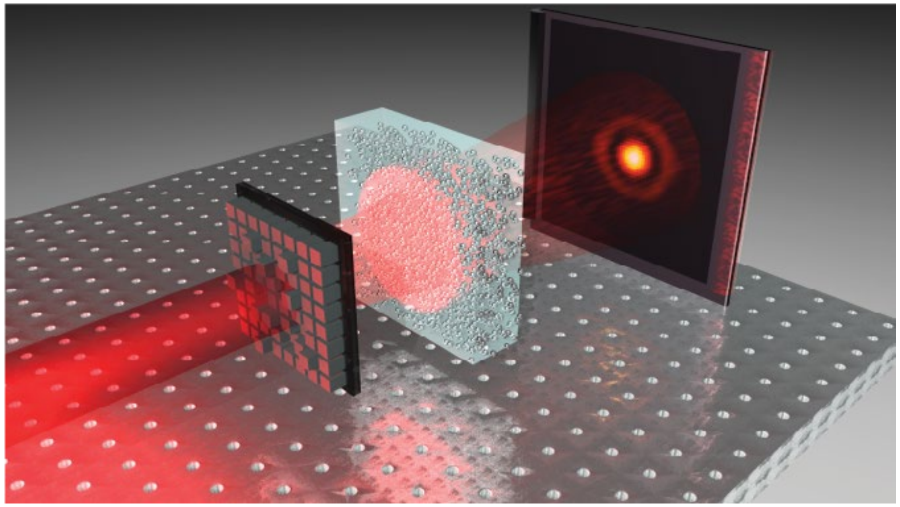
Speckle





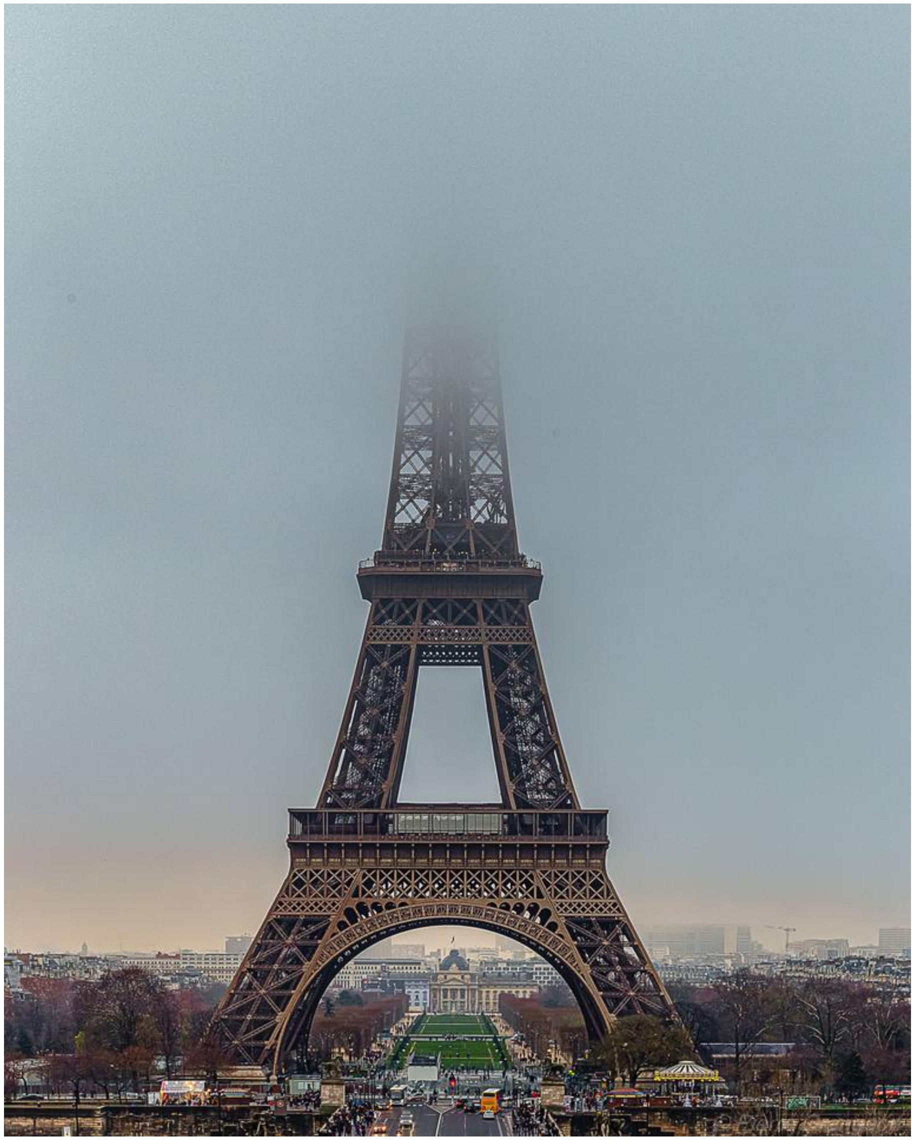
Wavefront shaping



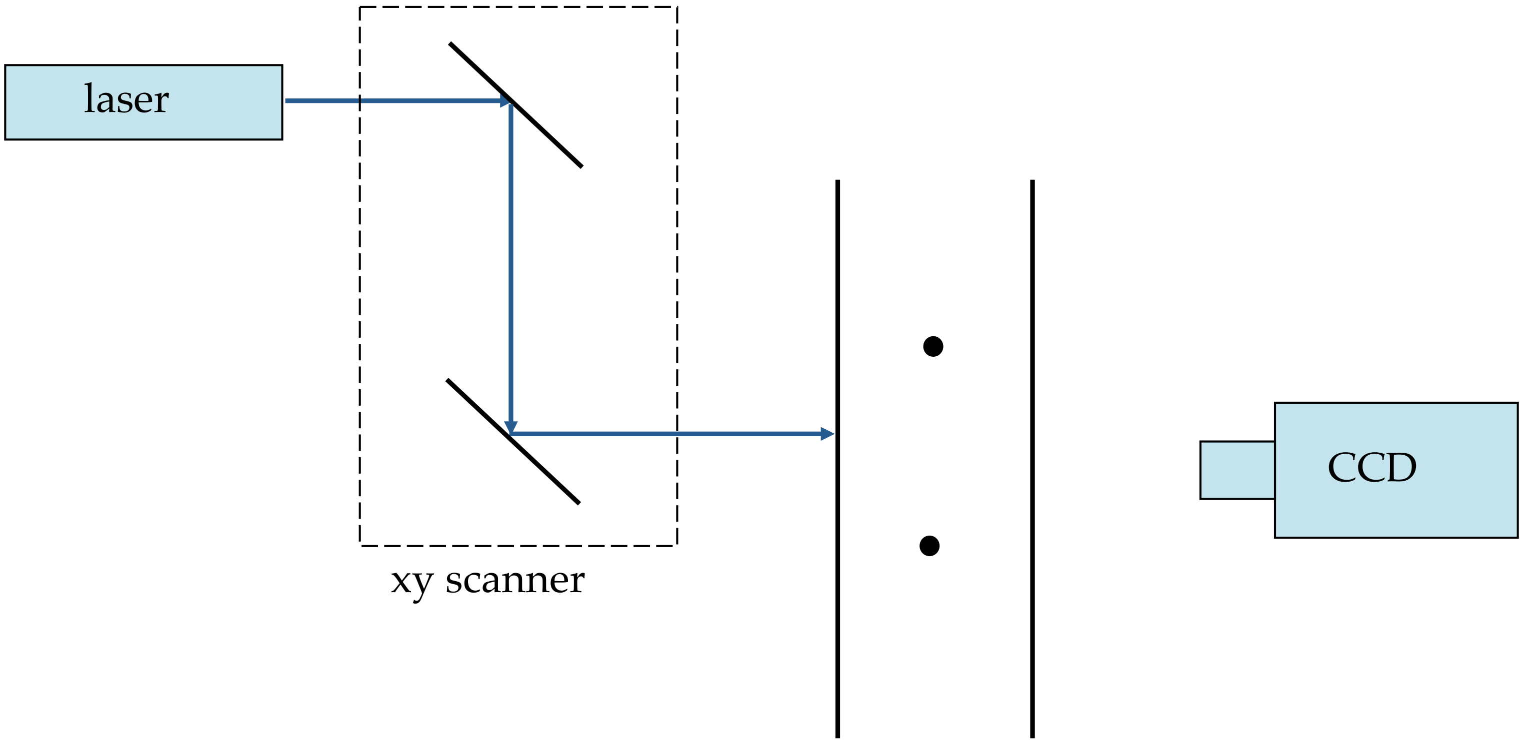


Cao, Mosk and Rotter, Nature Physics (2022)

Imaging

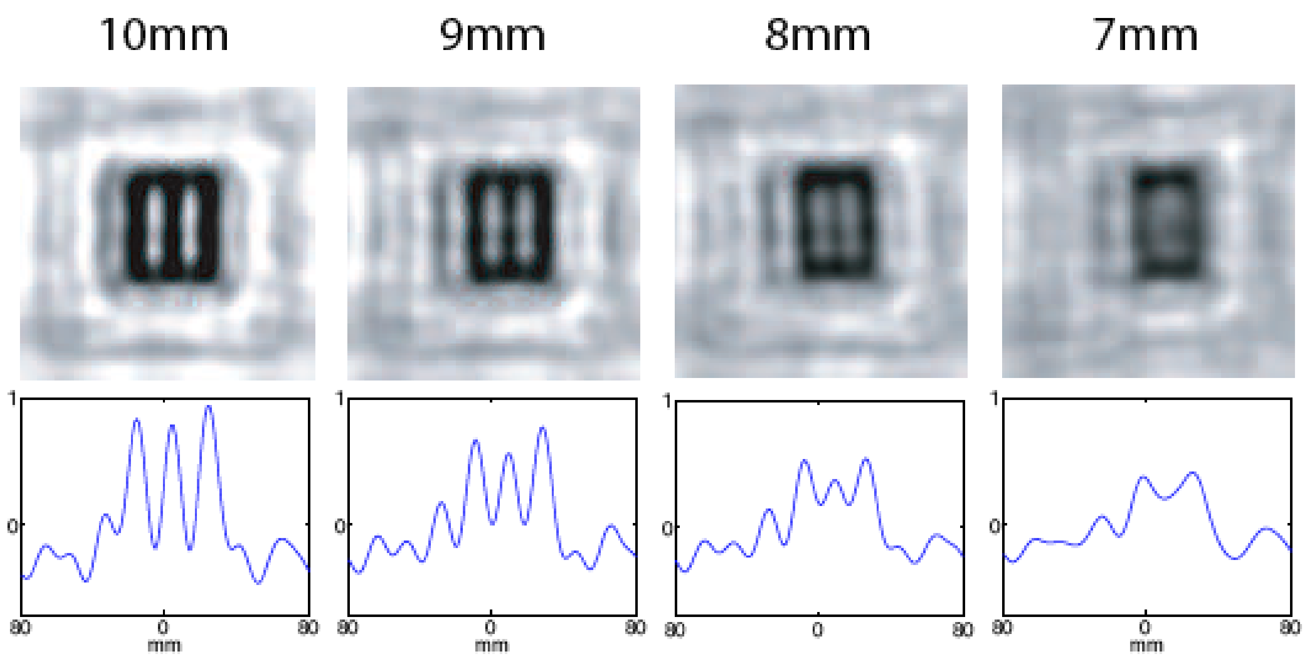


Optical tomography



Sample is of order $100\ell^*$ in size

Inverse problem



Scales

- Waves

  — scale of the wavelength λ

- Transport

  — scale of the scattering length ℓ_s

- Diffusion

  — macroscopic scale L

- Multiple scattering and separation of scales

$$\lambda \ll \ell_s \ll L$$

Waves, transport and diffusion

At microscopic scales, the field $u(\mathbf{x}, t)$ obeys the wave equation

$$\frac{\varepsilon(\mathbf{x})}{c^2} \frac{\partial^2 u}{\partial t^2} = \Delta u \ ,$$

where ε is the dielectric permittivity and for simplicity we do not consider polarization.

At mesoscopic scales, the specific intensity $I(\mathbf{x}, \hat{\mathbf{k}}, t)$ obeys the radiative transport equation

$$\frac{1}{c}\frac{\partial I}{\partial t} + \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} I = \frac{1}{\ell_s} \int d\hat{\mathbf{k}}' \left[A(\hat{\mathbf{k}}', \hat{\mathbf{k}}) I(\mathbf{x}, \hat{\mathbf{k}}') - A(\hat{\mathbf{k}}, \hat{\mathbf{k}}') I(\mathbf{x}, \hat{\mathbf{k}}) \right] \ .$$

The phase function A is normalized so that $\int A(\hat{\mathbf{k}}, \hat{\mathbf{k}}') d\hat{\mathbf{k}}' = 1$ for all $\hat{\mathbf{k}}$.

At macroscopic scales, the specific intensity is well approximated by the solution to the diffusion equation

$$\frac{1}{c}\frac{\partial U}{\partial t} = D\Delta U \ ,$$

where $I(\mathbf{x}, \hat{\mathbf{k}}, t) = U(\mathbf{x}, t) + \ell_s \hat{\mathbf{k}} \cdot \nabla U(\mathbf{x}, t)$ and $D = \frac{1}{3} c \ell_s$.

We view light propagation in disordered systems as the study of waves in random media. Averaging of randomness at the microscopic level leads to deterministic behavior at the macroscopic level.

Waves to transport

The RTE can be derived from the high-frequency behavior of wave propagation in random media. Field correlations (Wigner transform) are related to the specific intensity.

This is a long story. See R. Carminati and J. Schotland, *Principles of Scattering and Transport of Light* (Cambridge University Press, 2021).

The wave equation is invariant under time reversal but the transport equation is not. This is reminiscent of the problem of deriving kinetic equations from Hamiltonian mechanics.

Averaging over the random medium leads to the loss of time-reversal invariance.

This is a fantastic book for those of us who work on mathematical modeling of wave interaction with complex systems. […] It will undoubtedly become an indispensable aid to researchers in optical physics and optical engineering, and to anyone who wishes to move into the field.
Habib Ammari, ETH Zürich

This beautiful masterwork of Carminati and Schotland takes us from the foundational laws of physics expressed in Maxwell's equations through to the most quotidian of observable phenomena: light viewed through a murky liquid. […] Every step of the way is clearly explained and accessible. I suspect the reader may arrive looking for a particular chapter and find the whole book irresistible.
P. Scott Carney, The Institute of Optics, University of Rochester

Light scattering is one of the most well-studied phenomena in nature. It occupies a central place in optical physics and plays a key role in multiple fields of science and engineering. This volume presents a comprehensive introduction to the subject. For the first time, the authors bring together in a self-contained and systematic manner the physical concepts and mathematical tools that are used in the modern theory of light scattering and transport, presenting them in a clear, accessible way. The power of these tools is demonstrated by a framework that links various aspects of the subject: scattering theory to radiative transport, radiative transport to diffusion, and field correlations to the statistics of speckle patterns. For graduate students and researchers in optical physics and optical engineering, this book is an invaluable resource on the interaction of light with complex media and the theory of light scattering in disordered and complex systems.

Rémi Carminati is Professor of Physics at ESPCI Paris, before which he held a faculty position at École Centrale Paris. He was awarded the Fabry-de Gramont prize of the French Optical Society and is a Fellow of the Optical Society of America.

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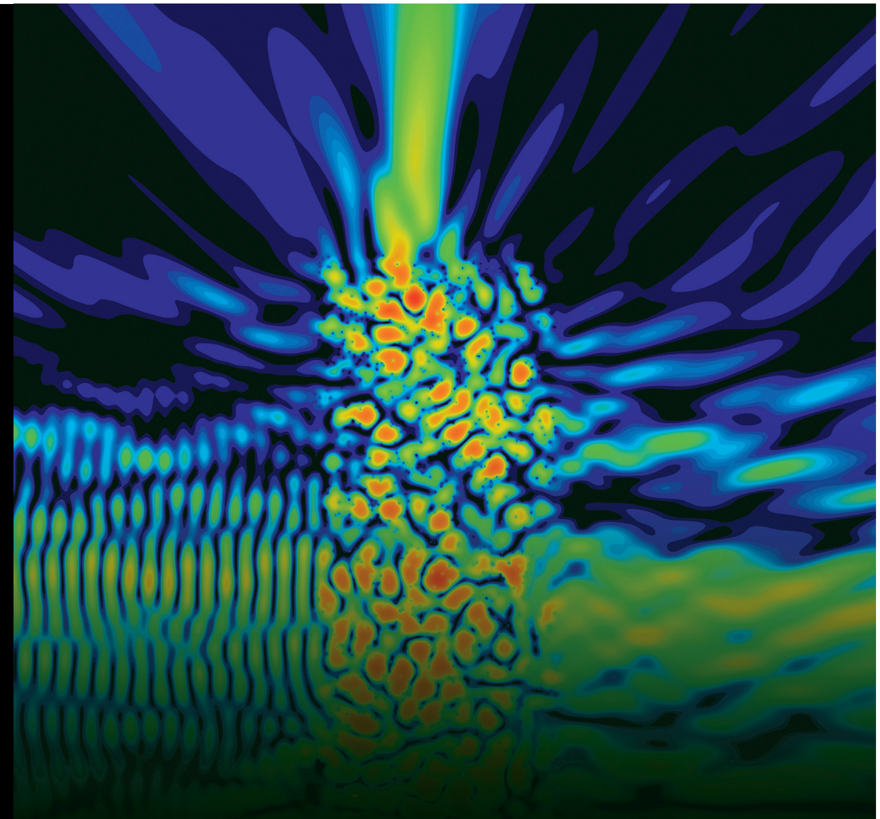
Carminati and Schotland

PRINCIPLES OF SCATTERING AND TRANSPORT OF LIGHT

CAMBRIDGE



PRINCIPLES OF SCATTERING AND TRANSPORT OF LIGHT

Rémi Carminati and John C. Schotland

Waves in random media

- Coherent backscattering
- Speckle correlations
- Localization
- Near-field effects
- Imaging

Quantum Optics

Quantum optics and random media

- Radiative transport is based on classical theories of light propagation

- Are there quantum effects in radiative transport?

  — spontaneous emission of single photons

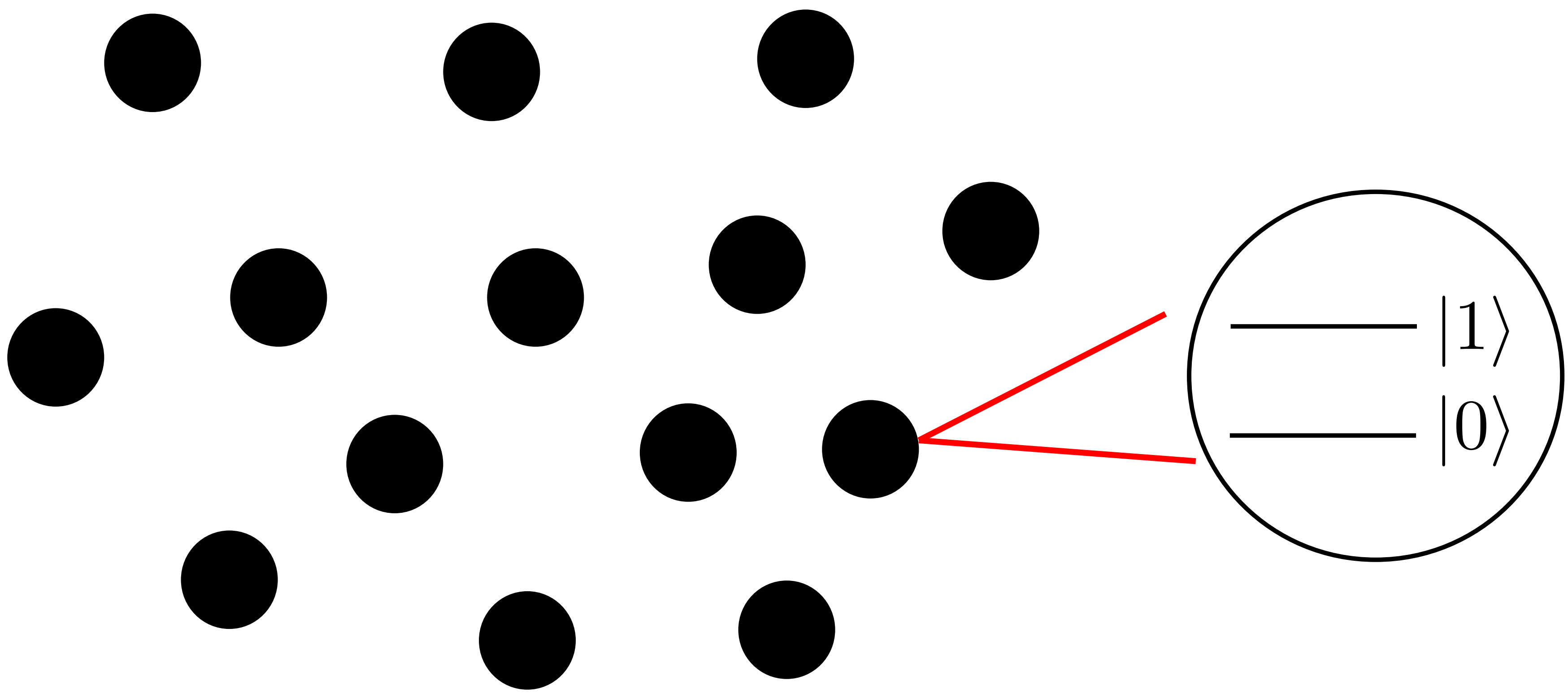
  — transport of entangled two-photon states

- New mathematical tools

  — many-body problem

  — nonlocal PDEs with random coefficients

Physical ideas



Quantization of the field

We consider a scalar model of the electromagnetic field (without polarization). The field u obeys the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \ .$$

We expand the solution into Fourier modes of the form

$$u(\mathbf{x}, t) = \sum_{\mathbf{k}} u_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

and find that

$$\ddot{u}_{\mathbf{k}} + \omega_k^2 u_{\mathbf{k}} = 0 \ .$$

This corresponds to independent harmonic oscillator modes with frequency $\omega_k = c|\mathbf{k}|$.

The oscillators are quantized by promoting the $u_{\mathbf{k}}$ to operators in the usual manner. The Hamiltonian of the quantized field is given by

$$H = \sum_{\mathbf{k}} \hbar\omega_k a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \ .$$

The creation and annihilation operators obey the bosonic commutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}] = \delta_{\mathbf{k}\mathbf{k}'} \ , \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0 \ .$$

Here $a_{\mathbf{k}}^{\dagger}$ and $a_{\mathbf{k}}$ are defined by

$$a_{\mathbf{k}}^{\dagger} |n_k\rangle = \sqrt{n_k + 1}\, |n_k + 1\rangle \ , \quad a_{\mathbf{k}} |n_k\rangle = \sqrt{n_k}\, |n_k - 1\rangle \ .$$

Photons are collective excitations of the quantized field. There are n_k photons in the state $|n_k\rangle$, each with energy $\hbar\omega_k$.

Two-level atom

We consider a two level atom with Hamiltonian

$$H_A = \hbar\Omega\sigma^\dagger\sigma \ ,$$

where Ω is the transition frequency of the atom. Here σ is the lowering operator and $\sigma^\dagger$ is the raising operator for the atomic states. That is, $\sigma = |0\rangle\langle 1|$, where $|0\rangle$ is the ground state and $|1\rangle$ is the excited state. Note that σ obeys the fermionic anticommutation relation $\{\sigma, \sigma^\dagger\} = 1$.

If the atom is initially in its excited state it will remain there forever; it is an eigenstate.

The atom has an electric dipole moment which couples to the field according to the interaction Hamiltonian

$$H_I = \hbar g \sum_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger} \sigma + a_{\mathbf{k}} \sigma^{\dagger} \right) \;,$$

where the coupling constant g is proportional to the dipole moment. The first term corresponds to loss of a photon by the atom and the gain of a photon by the field; the second term has the opposite effect.

The coupling to the field causes the excited state to decay; this is called spontaneous emisssion.

Many atoms

Consider the following Hamiltonian:

$$H = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \sum_{j} \hbar\Omega\sigma_j^{\dagger}\sigma_j + \hbar g \sum_{j}\sum_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger}\sigma_j e^{i\mathbf{k}\cdot\mathbf{x}_j} + a_{\mathbf{k}}\sigma_j^{\dagger} e^{-i\mathbf{k}\cdot\mathbf{x}_j} \right) \ ,$$

where $\sigma_j = |0_j\rangle\langle 1_j|$. The atoms interact through their coupling to the field.

According to the Schrodinger equation,

$$i\hbar\partial_t |\psi\rangle = H |\psi\rangle \, ,$$

the single excitation state

$$|\psi(t)\rangle = \sum_{j}\sum_{\mathbf{k}} \left(\alpha_{\mathbf{k}}(t) a_{\mathbf{k}}^{\dagger} + \beta_j(t)\sigma_j^{\dagger} \right) |0\rangle$$

evolves according to

$$
\begin{aligned}
i\frac{d}{dt}\alpha_{\mathbf{k}} &= \omega_{\mathbf{k}}\alpha_{\mathbf{k}} + g\sum_{j}\beta_j e^{-i\mathbf{k}\cdot\mathbf{x}_j} \\
i\frac{d}{dt}\beta_j &= \Omega\beta_j + g\sum_{\mathbf{k}}\alpha_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}_j} \quad .
\end{aligned}
$$

The number of excitations is conserved.

Spontaneous emission

Consider a single atom, which is initially in its excited state. If $g = 0$ (no coupling to the field), then

$$\beta(t) = e^{-i\Omega t} \ ,$$

and the atom remains in its excited state for all times.

If $g \neq 0$, the atom decays to its ground state due to coupling to the vacuum field. The probability that the atom is in its excited state is

$$P(t) = |\beta(t)|^2 = e^{-\gamma t} \ ,$$

where $\gamma = g^2\Omega^2/c^3$ is the decay rate.

This result is due to Wigner and Weisskopf (1930). It makes use of an approximation that breaks down at long times.

Many atom dynamics

When there are many atoms, the spontaneous emission of light becomes cooperative.

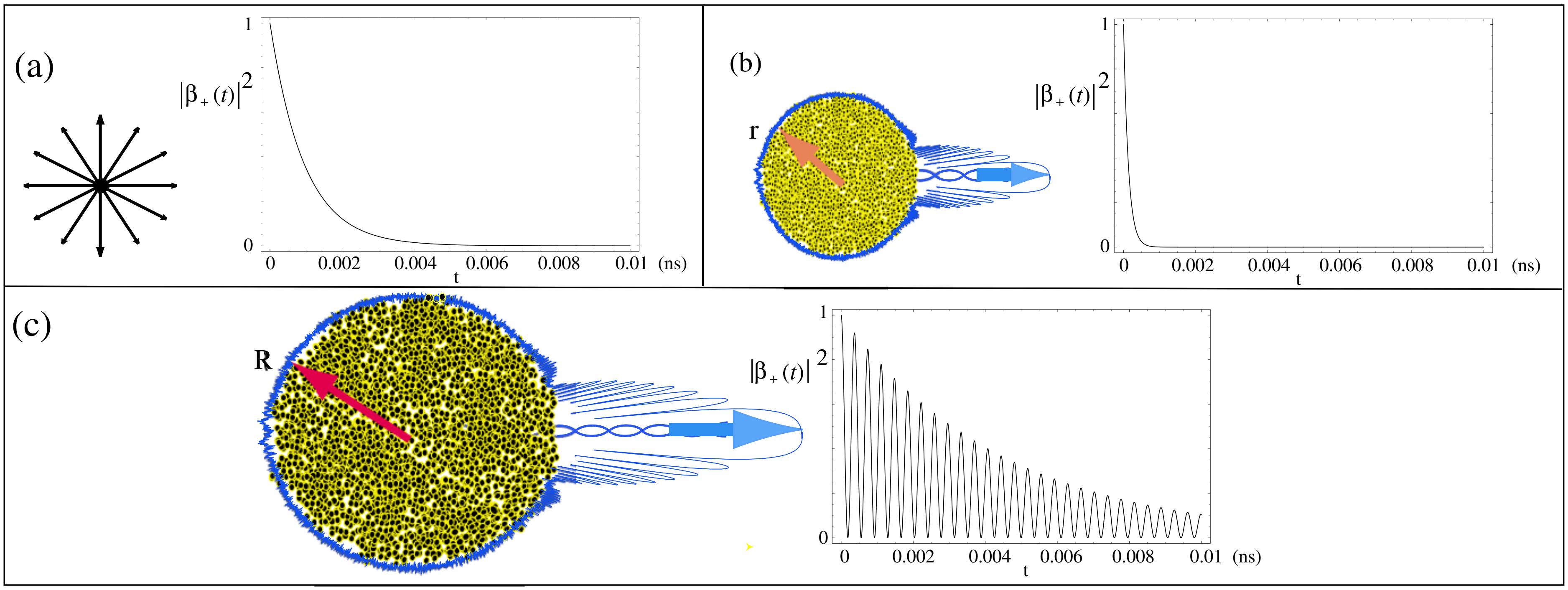
$$i\frac{d}{dt}\alpha_{\mathbf{k}} = \omega_{\mathbf{k}}\alpha_{\mathbf{k}} + g\sum_j \beta_j e^{-i\mathbf{k}\cdot\mathbf{x}_j}$$

$$i\frac{d}{dt}\beta_j = \Omega\beta_j + g\sum_{\mathbf{k}} \alpha_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}_j} \ .$$

The dynamics depends on the size of the system. For small systems, there is exponential decay at the rate $N\gamma$, where N is the number of atoms. For large systems there are oscillations and decay.

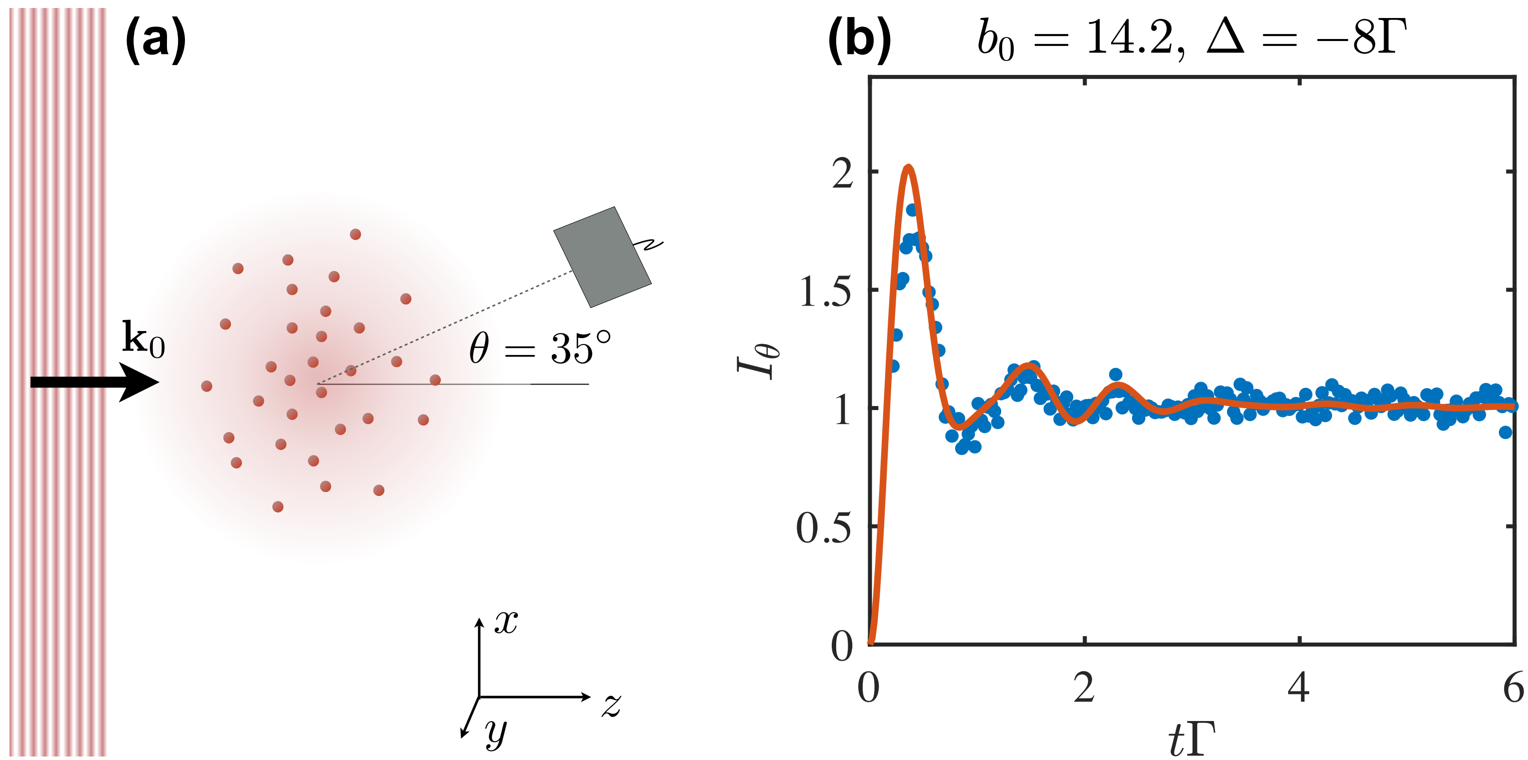
This is a computationally challenging problem.

Single-photon superradiance



Svidzinsky, Chang and Scully, Phys. Rev. Lett. (2008)

Cold trapped ^{87}Rb



Santo, Weiss, Cipris, Kaiser, Guerin, Bachelard and Schachenmayer, Phys. Rev A (2020)

Real-space quantization

The analysis of many-body problems requires new mathematical tools.

In order to treat the atoms and the field on the same footing, we introduce a real-space quantization of the fields.

The Hamiltonian of the field is given by

$$H = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} \ ,$$

where $\omega_{\mathbf{k}} = c|\mathbf{k}|$.

Let $\phi(\mathbf{x})$ denote the Fourier transform of $a_{\mathbf{k}}$,

$$\phi(\mathbf{x}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} \ .$$

Evidently, ϕ obeys the commutation relations $[\phi(\mathbf{x}), \phi^{\dagger}(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}')$ and $[\phi(\mathbf{x}), \phi(\mathbf{x}')] = 0$.

The Hamiltonian becomes

$$H = \hbar c \int d\mathbf{x} (-\Delta)^{1/2} \phi^{\dagger}(\mathbf{x}) \phi(\mathbf{x}) \ ,$$

since $|\mathbf{k}|$ is the Fourier multiplier of $(-\Delta)^{1/2}$.

The operator $(-\Delta)^{1/2}$ is non-local and is defined by the Fourier integral

$$(-\Delta)^{1/2} f(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} |\mathbf{k}| \tilde{f}(\mathbf{k}) \ .$$

It also has the spatial representation

$$(-\Delta)^{1/2} f(\mathbf{x}) = C \int d\mathbf{y} \, \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^4} \ ,$$

which is a singular integral.

The Hamiltonian now becomes

$$H = \hbar \int d\mathbf{x} \Big[c(-\Delta)^{1/2} \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) + \Omega\rho(\mathbf{x})\sigma^\dagger(\mathbf{x})\sigma(\mathbf{x}) + g\rho(\mathbf{x}) \left(\phi^\dagger(\mathbf{x})\sigma(\mathbf{x}) + \phi(\mathbf{x})\sigma^\dagger(\mathbf{x}) \right) \Big] ,$$

where $\sigma(\mathbf{x})$ is the atomic lowering operator and $\rho(\mathbf{x})$ is the number density of atoms.

Consider a single-excitation state of the form

$$|\mathbf{\Psi}\rangle = \int d\mathbf{x} \left[\psi(\mathbf{x},t)\phi^\dagger(\mathbf{x}) + \rho(\mathbf{x})a(\mathbf{x},t)\sigma^\dagger(\mathbf{x}) \right] |0\rangle ,$$

where $|0\rangle$ is the combined vacuum state of the field and the ground state of the atoms. Here $a(\mathbf{x},t)$ denotes the probability amplitude for exciting an atom and $\psi(\mathbf{x},t)$ is the amplitude for creating a photon.

The quantity $|\psi(\mathbf{x},t)|^2$ is proportional to the number of photons registered by a detector.

The dynamics of the state $|\mathbf{\Psi}\rangle$ is governed by the Schrodinger equation

$$i\hbar\partial_t|\mathbf{\Psi}\rangle = H|\mathbf{\Psi}\rangle \ .$$

We find that a and ψ obey the nonlocal PDEs

$$\begin{aligned} i\partial_t\psi &= c(-\Delta)^{1/2}\psi + g\rho(\mathbf{x})a \ , \\ i\partial_t a &= g\psi + \Omega a \ . \end{aligned}$$

The amplitudes obey the normalization condition

$$\int d\mathbf{x}\left(|\psi(\mathbf{x},t)|^2 + \rho(\mathbf{x})|a(\mathbf{x},t)|^2\right) = 1 \ ,$$

which guarantees conservation of probability.

Spontaneous emission

Consider a single atom with $\rho(\mathbf{x}) = \delta(\mathbf{x})$. We assume that the atom is initially in its excited state and that there are no photons present in the field. The probability the atom is in its excited state decays exponentially:

$$|a(0,t)|^2 = e^{-\gamma t} \ ,$$

where

$$\gamma = \frac{g^2 \Omega^2}{\pi c^3} \ .$$

Here we make use of a pole approximation and recover the result of Wigner and Weisskopf.

Constant density

Suppose that $\rho(\mathbf{x}) = \rho_0$. Then

$$i\partial_t \psi = c(-\Delta)^{1/2}\psi + g\rho_0 a \ ,$$
$$i\partial_t a = g\psi + \Omega a \ .$$

can be solved explicitly.

Suppose that initially there is a localized region of excited atoms. The initial amplitudes are taken to be

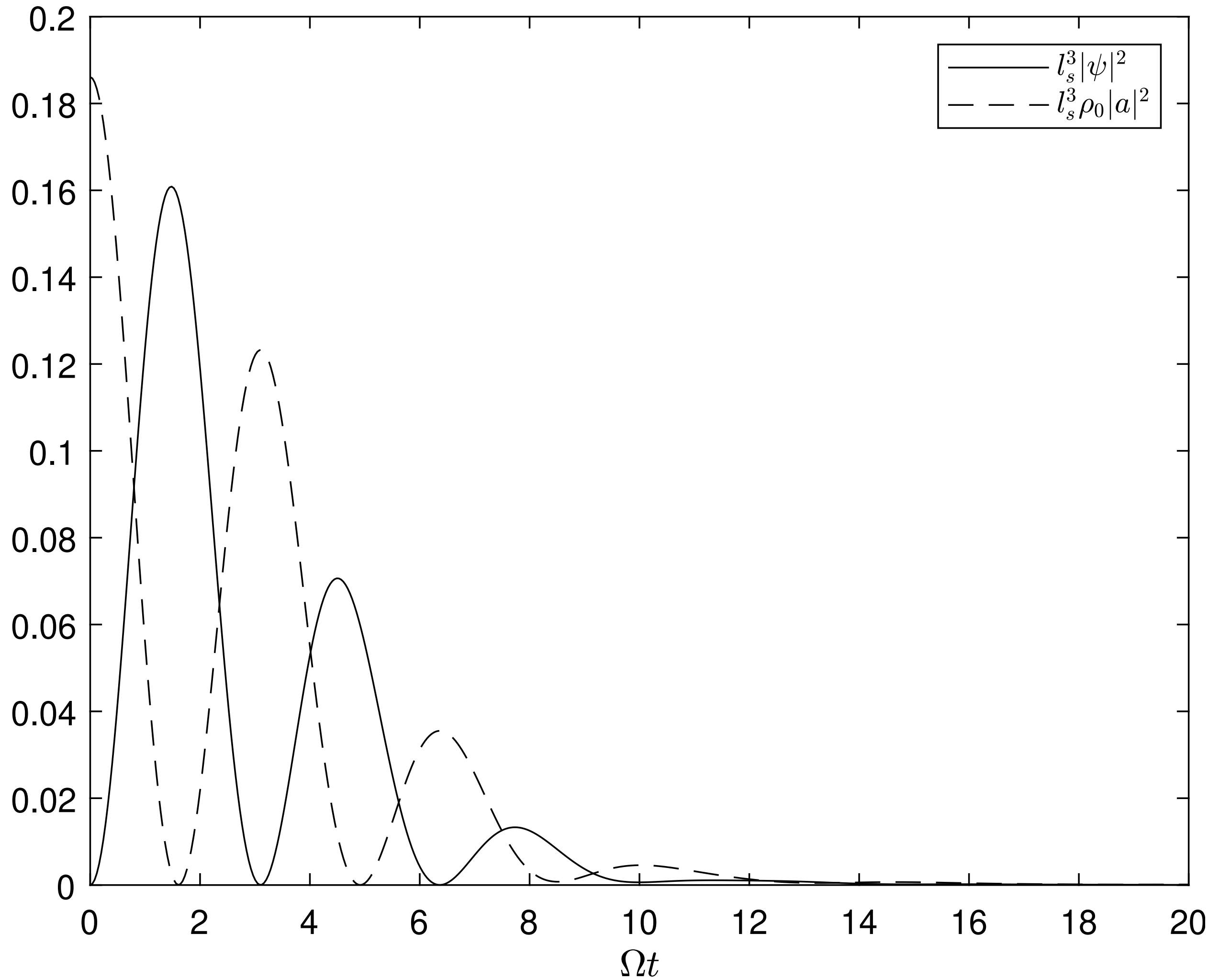
$$\psi(\mathbf{x}, 0) = 0 \ ,$$
$$\sqrt{\rho_0}\, a(\mathbf{x}, 0) = \left(\frac{1}{\pi l_s^2}\right)^{3/4} e^{-|\mathbf{x}|^2/2l_s^2}.$$

$$\psi(\mathbf{x},t) = \frac{g\sqrt{2}\rho_0 l_s^{3/2}}{\pi^{3/2}|\mathbf{x}|}\int_0^\infty dk\, k\,\sin(k|\mathbf{x}|)\frac{e^{-i\lambda_+(k)t} - e^{-i\lambda_-(k)t}}{\lambda_+(k) - \lambda_-(k)} e^{-l_s^2 k^2/2}\,,$$

$$\sqrt{\rho_0}\,a(\mathbf{x},t) = \frac{\sqrt{2} l_s^{3/2}}{\pi^{3/2}|\mathbf{x}|}\int_0^\infty dk\, k\,\sin(k|\mathbf{x}|) \times \frac{(\lambda_+(k) - \Omega)e^{-i\lambda_-(k)t} - (\lambda_-(k) - \Omega)e^{-i\lambda_+(k)t}}{\lambda_+(k) - \lambda_-(k)} e^{-l_s^2 k^2/2}\,,$$

where

$$\lambda_\pm(\mathbf{k}) = \frac{c|\mathbf{k}| + \Omega \pm \sqrt{(c|\mathbf{k}| - \Omega)^2 + 4g^2\rho_0}}{2}\,.$$



$\Omega l_s/c = 1$, $\rho_0(g/\Omega)^2 = 1$ and $|\mathbf{x}| = l_s$

Many atoms

Let $\rho(\mathbf{x}) = \sum_{j=1}^{N} \delta(\mathbf{x} - \mathbf{x}_j)$. Then as $t \to \infty$,

$$a(t) = \sum_j R_j e^{ip_j t} a(0) + \sum_j S_j e^{-z_j t} a(0) - \frac{g^2 e^{i\Omega t}}{2\pi^2 i c \Omega^2 t^3} a(0) + O\left(\frac{1}{t^4}\right) ,$$

where p_j and z_j are poles of two matrix-valued functions, and R_j and S_j are corresponding projectors.

Note that there is a crossover from exponential to algebraic decay at long times.

Random media

In a random medium, we wish to determine $\langle|a(\mathbf{x},t)|^2\rangle$ and $\langle|\psi(\mathbf{x},t)|^2\rangle$, where $\langle\cdots\rangle$ denotes statistical averaging.

$$\begin{aligned} i\partial_t\psi &= c(-\Delta)^{1/2}\psi + g\rho(\mathbf{x})a \ , \\ i\partial_t a &= g\psi + \Omega a \ . \end{aligned}$$

The atomic density $\rho(\mathbf{x})$ is taken to be a random field of the form $\rho(\mathbf{x}) = \rho_0(1+\eta(\mathbf{x}))$, where ρ_0 is constant. The density fluctuation η is a statistically homogeneous and isotropic random field with correlations

$$\begin{aligned} \langle\eta(\mathbf{x})\rangle &= 0 \ , \\ \langle\eta(\mathbf{x})\eta(\mathbf{y})\rangle &= C(|\mathbf{x}-\mathbf{y}|) \ . \end{aligned}$$

The solutions a and ψ oscillate rapidly on the scale of the wavelength. We are interested in the high-frequency regime where the propagation distance is long compared to the wavelength, the propagation time is large compared to the period, and ρ is slowly varying.

The high-frequency, weak disorder regime is precisely the setting in which radiative transport theory holds for classical wave fields.

The analog of the RTE can be derived from the asymptotics of the Wigner transform.

The Wigner transform is defined by

$$W_{ij}(\mathbf{x},\mathbf{k},t) = \int \frac{d\mathbf{x}'}{(2\pi)^n} e^{i\mathbf{k}\cdot\mathbf{x}'} \Phi_i(\mathbf{x}-\mathbf{x}'/2,t)\Phi_j^*(\mathbf{x}+\mathbf{x}'/2,t) \ ,$$

where $\Phi(\mathbf{x},t) = (\psi(\mathbf{x},t), a(\mathbf{x},t))$.

The probability densities $|\psi(\mathbf{x},t)|^2$ and $|a(\mathbf{x},t)|^2$ are related to the Wigner transform by

$$|\psi(\mathbf{x},t)|^2 = \int d\mathbf{k} W_{11}(\mathbf{x},\mathbf{k},t) \ ,$$
$$|a(\mathbf{x},t)|^2 = \int d\mathbf{k} W_{22}(\mathbf{x},\mathbf{k},t) \ .$$

The diagonal elements of W are real-valued, but not generally nonnegative. However, in the high-frequency limit they become nonnegative.

The average Wigner transform can be decomposed into modes as

$$\langle W(\mathbf{x},\mathbf{k},t)\rangle = a_+(\mathbf{x},\mathbf{k},t)\mathbf{b}_+(\mathbf{k})\mathbf{b}_+^T(\mathbf{k}) + a_-(\mathbf{x},\mathbf{k},t)\mathbf{b}_-(\mathbf{k})\mathbf{b}_-^T(\mathbf{k}) \ ,$$

$$\mathbf{b}_\pm(\mathbf{k}) = \frac{1}{\sqrt{(\lambda_\pm(\mathbf{k})-\Omega)^2+g^2\rho_0}}\begin{bmatrix}\lambda_\pm(\mathbf{k})-\Omega \\ g\sqrt{\rho_0}\end{bmatrix}.$$

$$\lambda_\pm(\mathbf{k}) = \frac{c|\mathbf{k}|+\Omega \pm \sqrt{(c|\mathbf{k}|-\Omega)^2+4g^2\rho_0}}{2} \ .$$

In the high-frequency limit, the modes $a_\pm$ obey a kinetic equation of the form

$$\frac{1}{c}\partial_t a_\pm(\mathbf{x},\mathbf{k},t) + f_\pm(\mathbf{k})\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}} a_\pm(\mathbf{x},\mathbf{k},t) = \sigma_\pm(\mathbf{k})\int d\hat{\mathbf{k}}' \left[A(\mathbf{k}',\mathbf{k})a_\pm(\mathbf{x},\mathbf{k}',t) - A(\mathbf{k},\mathbf{k}')a_\pm(\mathbf{x},\mathbf{k},t)\right] \ ,$$

The coefficients are given in terms of the correlation function of the disorder:

$$\sigma_\pm(\mathbf{k}) = \frac{4\pi(g^2\rho_0)^2|\lambda_\pm(\mathbf{k})-\Omega|}{c^2((\lambda_\pm(\mathbf{k})-\Omega)^2+g^2\rho_0)^2}\sqrt{(c|\mathbf{k}|-\Omega)^2+4g^2\rho_0}|\mathbf{k}|^2$$
$$\times\int\frac{d\widehat{\mathbf{k}}'}{(2\pi)^3}\,\tilde{C}(|\mathbf{k}|(\widehat{\mathbf{k}}-\widehat{\mathbf{k}}'))\ ,$$
$$A(\mathbf{k},\mathbf{k}') = \frac{\tilde{C}(|\mathbf{k}|(\widehat{\mathbf{k}}-\widehat{\mathbf{k}}'))}{\int d\widehat{\mathbf{k}}'\tilde{C}(|\mathbf{k}|(\widehat{\mathbf{k}}-\widehat{\mathbf{k}}'))}\ ,$$
$$f_\pm(\mathbf{k}) = \frac{(\lambda_\pm(\mathbf{k})-\Omega)^2}{(\lambda_\pm(\mathbf{k})-\Omega)^2+g^2\rho_0}\ .$$

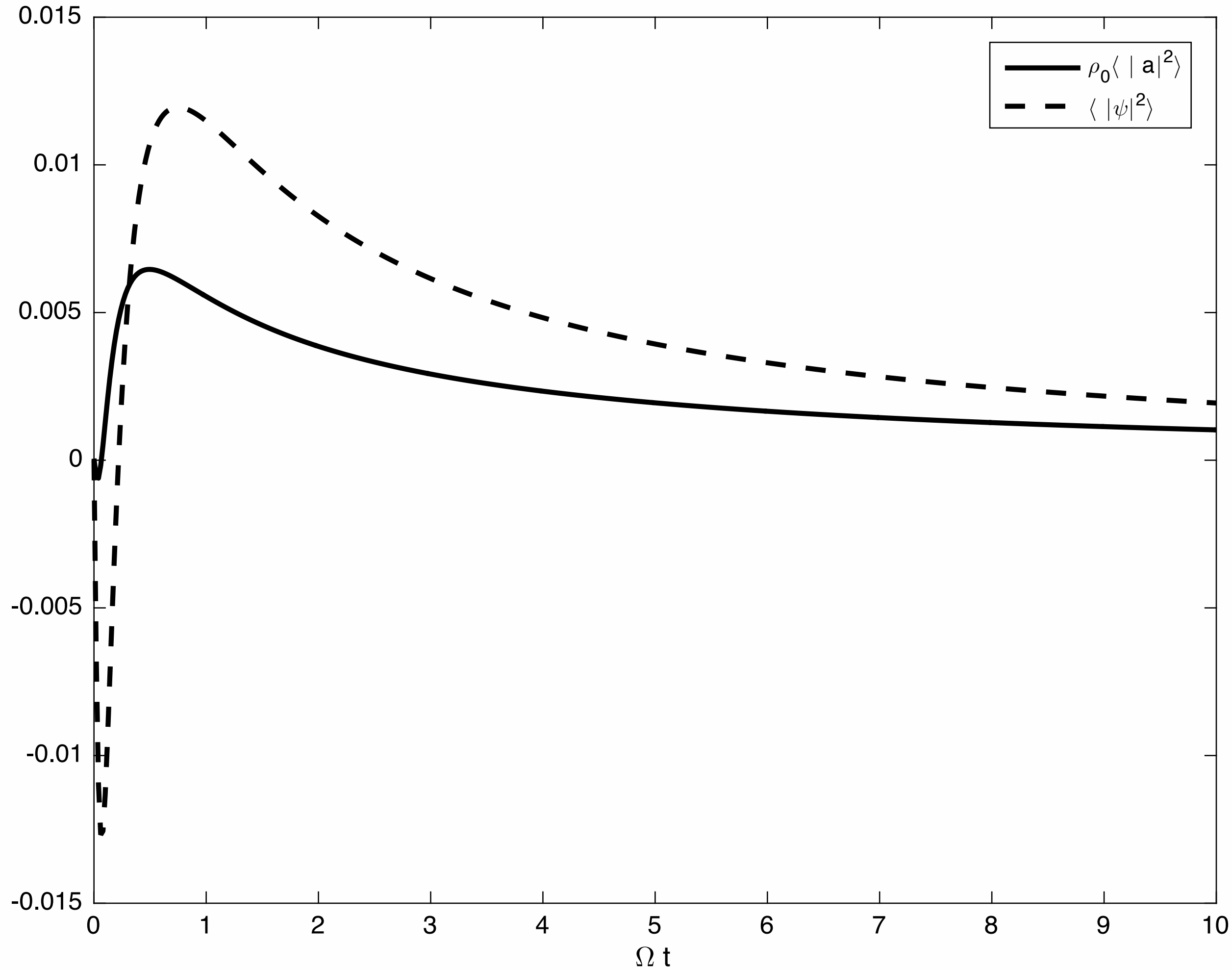
We suppose that the atoms are initially excited in a volume of linear dimensions l_s and that there are no photons present in the field:

$$a(\mathbf{x},0) = \left(\frac{1}{\pi l_s^2}\right)^{3/4} e^{-|\mathbf{x}|^2/2l_s^2} \ ,$$
$$\psi(\mathbf{x},0) = 0 \ .$$

The kinetic equations are solved in the diffusion approximation for an infinite medium. We assume isotropic scattering with $A = 1/(4\pi)$ and set $\Omega l_s/c = 1$, $\rho_0 (g/\Omega)^2 = 1$.

At long times ($\Omega t \gg 1$)

$$\langle |a(\mathbf{x},t)|^2 \rangle = \frac{C_1}{t^{3/2}} + O\left(\frac{1}{t^{5/2}}\right) \ ,$$
$$\langle |\psi(\mathbf{x},t)|^2 \rangle = \frac{C_2}{t^{3/2}} + O\left(\frac{1}{t^{5/2}}\right) \ .$$



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Ω t
$\rho_0\langle\, |a|^2\rangle$
$\langle\, |\psi|^2\rangle$

Two-photon states

The physics of two-photon states is much richer. There can be entanglement of both the photons and the atoms.

Consider a two-photon state of the form

$$|\mathbf{\Psi}\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \Big[\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)\phi^\dagger(\mathbf{x}_1)\phi^\dagger(\mathbf{x}_2) + \rho(\mathbf{x}_1)\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)\phi^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2) + \rho(\mathbf{x}_1)\rho(\mathbf{x}_2)a(\mathbf{x}_1,\mathbf{x}_2,t)\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2)\Big]|0\rangle \ .$$

Here a denotes the amplitude for exciting two atoms, ψ_2 is the amplitude for creating two photons, and ψ_1 is the amplitude for jointly exciting an atom and creating a photon.

The dynamics of the state $|\mathbf{\Psi}\rangle$ is governed by the Schrodinger equation

$$i\hbar\partial_t|\mathbf{\Psi}\rangle = H|\mathbf{\Psi}\rangle \ .$$

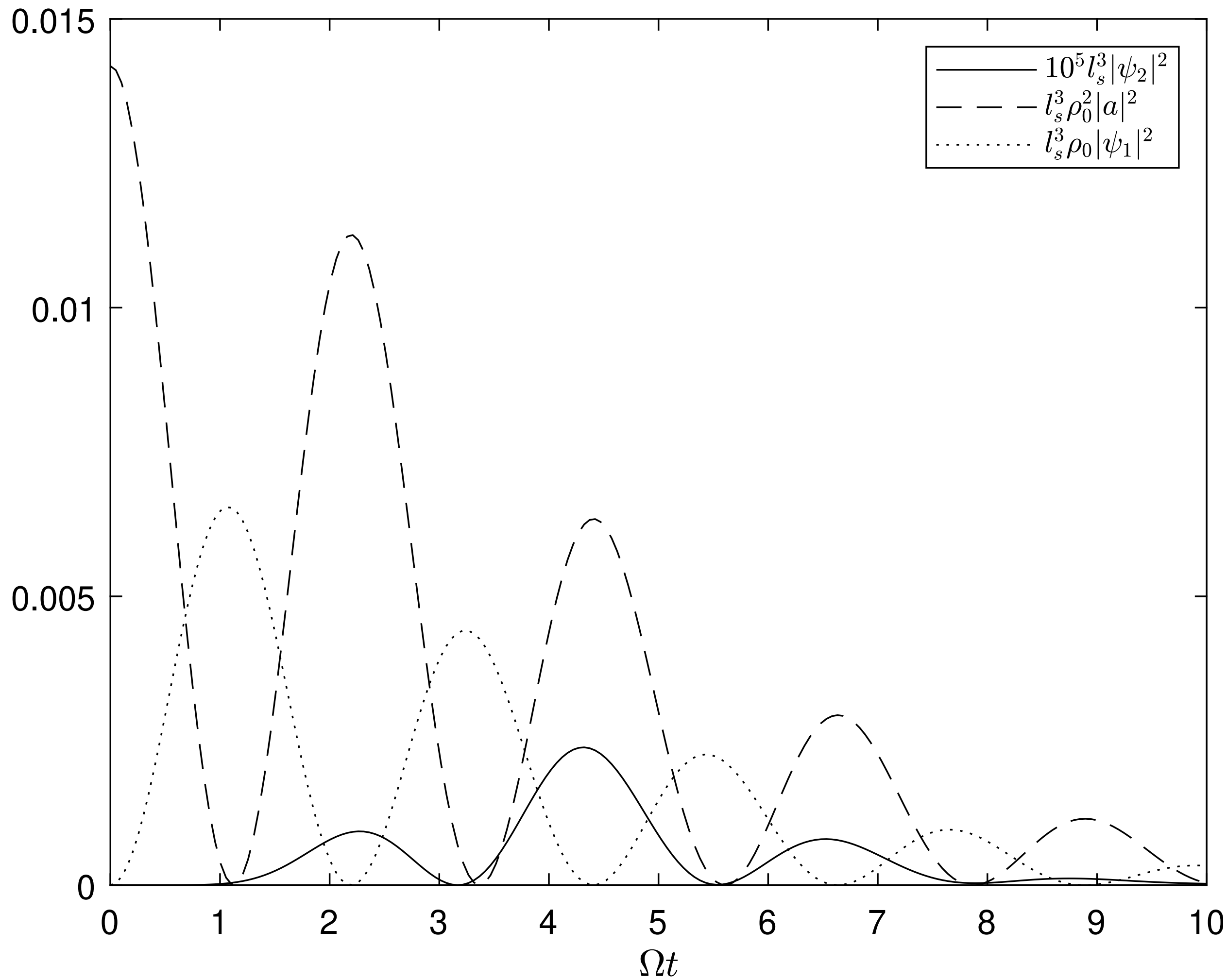
We find that a, ψ_1 and ψ_2 obey

$$\begin{aligned} i\partial_t\psi_2 &= c(-\Delta_{\mathrm{x}_1})^{1/2}\psi_2 + c(-\Delta_{\mathrm{x}_2})^{1/2}\psi_2 + \frac{g}{2}\left(\rho(\mathrm{x}_1)\psi_1 + \rho(\mathrm{x}_2)\tilde{\psi}_1\right) \ , \\ i\partial_t\psi_1 &= \left[c(-\Delta_{\mathrm{x}_2})^{1/2} + \Omega\right]\psi_1 - 2g\rho(\mathrm{x}_2)a + 2g\psi_2 \ , \\ i\partial_t a &= \frac{g}{2}(\tilde{\psi}_1 - \psi_1) + 2\Omega a \ , \end{aligned}$$

where $\tilde{\psi}_1(\mathrm{x}_1, \mathrm{x}_2) = \psi_1(\mathrm{x}_2, \mathrm{x}_1)$.

For the case of a single atom, we obtain a modified exponential decay of the population of the excited state, which describes the process of stimulated emission. Likewise the case of a medium with constant density can be analyzed.

Two photons constant density



Two-photon RTE

In a random medium, the average Wigner transform can be expanded into modes in a manner similar to the one-photon problem. It can be seen that in the high-frequency limit, the modes $a_i(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2)$, $i = 1, 2, 3, 4$, obey kinetic equations of the form

$$\frac{1}{c}\partial_t a_i + \hat{\mathbf{k}}_1 \cdot \nabla_{\mathbf{x}_1} a_i + \hat{\mathbf{k}}_2 \cdot \nabla_{\mathbf{x}_2} a_i = \mathcal{T}_i a_i \ .$$

The coefficients and the transport operator $\mathcal{T}_i$ are related to density correlations.

The diffusion approximation for a_i is constructed in the standard way.

We suppose that two photons are present in the field and that the atoms are initially in their ground states:

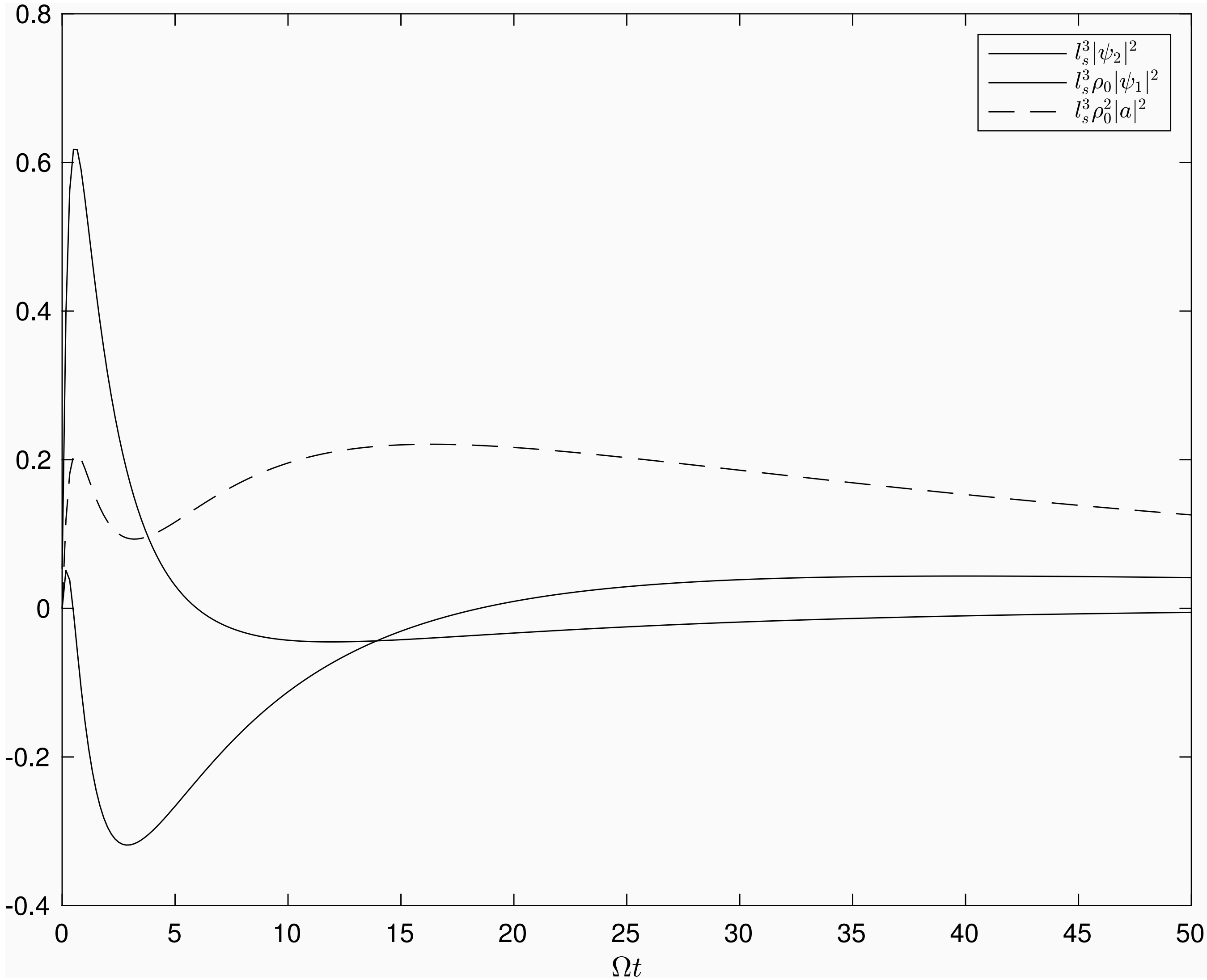
$$\psi_2(\mathbf{x}_1,\mathbf{x}_2,0) = C\left(e^{-|\mathbf{x}_1-\mathbf{y}_0|^2/2l_s^2}e^{-|\mathbf{x}_2-\mathbf{y}_1|^2/2l_s^2} + e^{-|\mathbf{x}_1-\mathbf{y}_1|^2/2l_s^2}e^{-|\mathbf{x}_2-\mathbf{y}_0|^2/2l_s^2}\right) ,$$
$$\psi_1(\mathbf{x}_1,\mathbf{x}_2,0) = 0 ,$$
$$a(\mathbf{x}_1,\mathbf{x}_2,0) = 0 .$$

We note that the initial two-photon state is entangled (not separable).

The kinetic equations are solved in the diffusion approximation.

At long times ($\Omega t \gg 1$)

$$\langle|\psi_1(\mathbf{x}_1,\mathbf{x}_2,t)|^2\rangle = \frac{C_1}{t^3} + O\left(\frac{1}{t^4}\right) ,$$
$$\langle|\psi_2(\mathbf{x}_1,\mathbf{x}_2,t)|^2\rangle = \frac{C_2}{t^3} + O\left(\frac{1}{t^4}\right) ,$$
$$\langle|a(\mathbf{x}_1,\mathbf{x}_2,t)|^2\rangle = \frac{C_3}{t^3} + O\left(\frac{1}{t^4}\right) .$$



$l_s^3|\psi_2|^2$
$l_s^3\rho_0|\psi_1|^2$
$l_s^3\rho_0^2|a|^2$
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
5
10
15
20
25
30
35
40
45
50
Ωt

Related topics

- Resonances and bound states
- Band structure and homogenization
- Anderson localization
- Solvers for nonlocal PDEs

Applications

- Imaging

  — quantum correlations

  — superresolution

- Communications

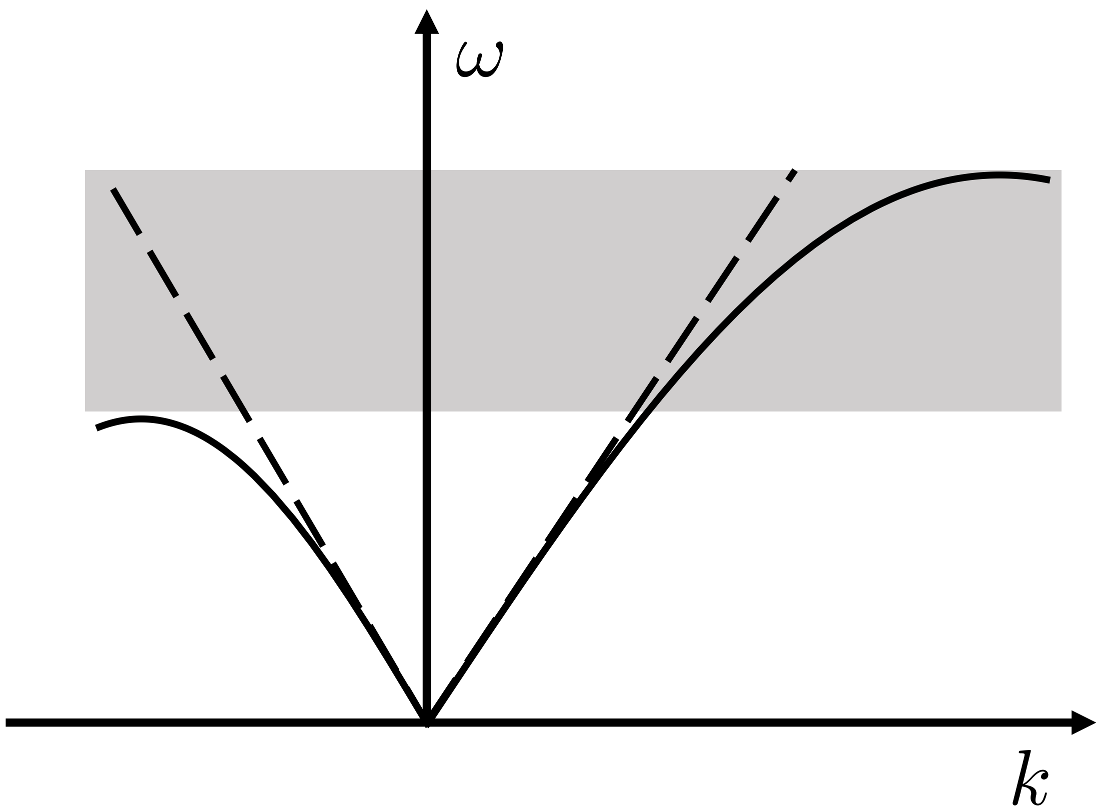
  — quantum information and disorder

  — channel capacity

One-way waveguides

Waveguides in which light propagates in one direction can serve as one-way carriers of quantum information.

The violation of reciprocity is due to an asymmetric dispersion relation.



Model

Consider the following Hamiltonian for a one-way waveguide containing a collection of two-level atoms:

$$H = \hbar \int dx \Big[iv\partial_x \phi^\dagger(x)\phi(x) + \Omega\rho(x)\sigma^\dagger(\mathbf{x})\sigma(\mathbf{x}) + g\rho(x)\left(\phi^\dagger(x)\sigma(x) + \phi(x)\sigma^\dagger(x)\right)\Big] \ .$$

The single excitation energy eigenstate $|\mathbf{\Psi}\rangle$ obeys the Schrodinger equation $H|\mathbf{\Psi}\rangle = \hbar\omega|\mathbf{\Psi}\rangle$, with

$$|\mathbf{\Psi}\rangle = \int dx \left[\psi(x)\phi^\dagger(x) + \rho(x)a(x)\sigma^\dagger(x)\right]|0\rangle \ ,$$

The amplitudes a and ψ satisfy

$$i\partial_x\psi + \gamma\rho(\mathbf{x})\psi = \frac{\omega}{v}\psi \ ,$$

$$a = \frac{(\omega - \Omega)}{g}\psi \ ,$$

where $\gamma = g^2/(\omega - \Omega)$.

Integration of the above gives

$$\psi(x) = \exp\left[-i\frac{\omega}{v}x + i\frac{\gamma}{v}\int_0^x \rho(y)dy\right]\psi(0) \ ,$$

Note that $|\psi(x)|^2$ is independent of $\rho(x)$. Transport is independent of configuration.

Waveguide arrays

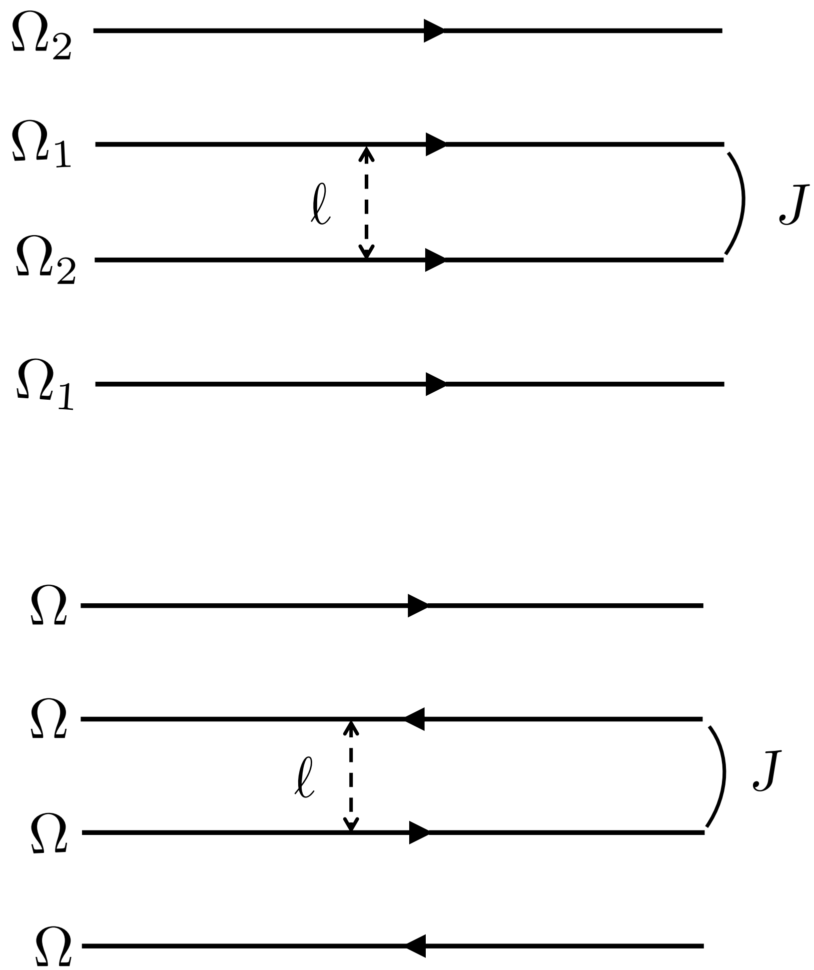
We consider the propagation of a single photon in an array of one-way waveguides.

The waveguides, each of which contains many two-level atoms, are arranged in a one-dimesional lattice.

In a chiral array, the atomic frequencies alternate in value between two interpenetrating sublattices.

In an antichiral array, the group velocities alternate in sign between the sublattices.

Chiral and antichiral arrays



The Hamiltonian of the system is $H = H_A + H_F + H_I$. Here the atomic Hamiltonian H_A is given by

$$H_A = \hbar\omega_0 \int dx \sum_n \rho_n(x)\sigma^\dagger(x)\sigma(x) \ .$$

The Hamiltonian of the optical field H_F is

$$H_F = \hbar \int dx \sum_n \Big[\phi_n^\dagger(x)\left(\Omega_n + iv_n\partial_x\right)\phi_n(x) \\ + J_0(\phi_n^\dagger(x)\phi_{n+1}(x) + \phi_{n+1}^\dagger(x)\phi_n(x))\Big] \ .$$

The interaction between the atoms and the field is of the form

$$H_I = \hbar g \int dx \sum_n \rho_n(x)\Big[\sigma^\dagger(x)\phi_n(x) + \sigma(x)\phi_n^\dagger(x)\Big] \ .$$

We suppose that the system is in a single-excitation state of the form

$$|\mathbf{\Psi}\rangle = \int dx \sum_n \left[a_n(x)\sigma^\dagger(x) + \psi_n(x)\phi_n^\dagger(x)\right]|0\rangle \ .$$

Making use of the Schrodinger equation $H|\mathbf{\Psi}\rangle = E_0|\mathbf{\Psi}\rangle$ we find that

$$iv_n\partial_x\psi_n + \Omega_n\psi_n + J_0\left(\psi_{n+1} + \psi_{n-1}\right) + \gamma\rho_n(x)\psi_n = E_0\psi_n \ ,$$

We split the above into even and odd parts and take the continuum limit. Let ψ_1 and ψ_2 denote the values of ψ in each of the sublattices. We then have

$$iv_1\partial_x\psi_1 + \Omega\psi_1 + iJ\partial_y\psi_2 + V(x,y)\psi_1 = E\psi_1 \ ,$$
$$iv_2\partial_x\psi_2 - \Omega\psi_2 + iJ\partial_y\psi_1 + V(x,y)\psi_2 = E\psi_2 \ .$$

where $\Omega = (\Omega_1 - \Omega_2)/2$ and $E = E_0 + (\Omega_1 + \Omega_2)/2$.

Dirac equations

For a chiral array, the frequencies differ in each sublattice and the group velocities are the same with $v_1 = v_2 = v$. We then find that $\psi = (\psi_1, \psi_2)$ obeys the Dirac equation

$$iv\partial_x\psi + iJ\alpha\partial_y\psi + \Omega\beta\psi + V(x,y)\psi = E\psi \ ,$$

where α and β are the Pauli matrices

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

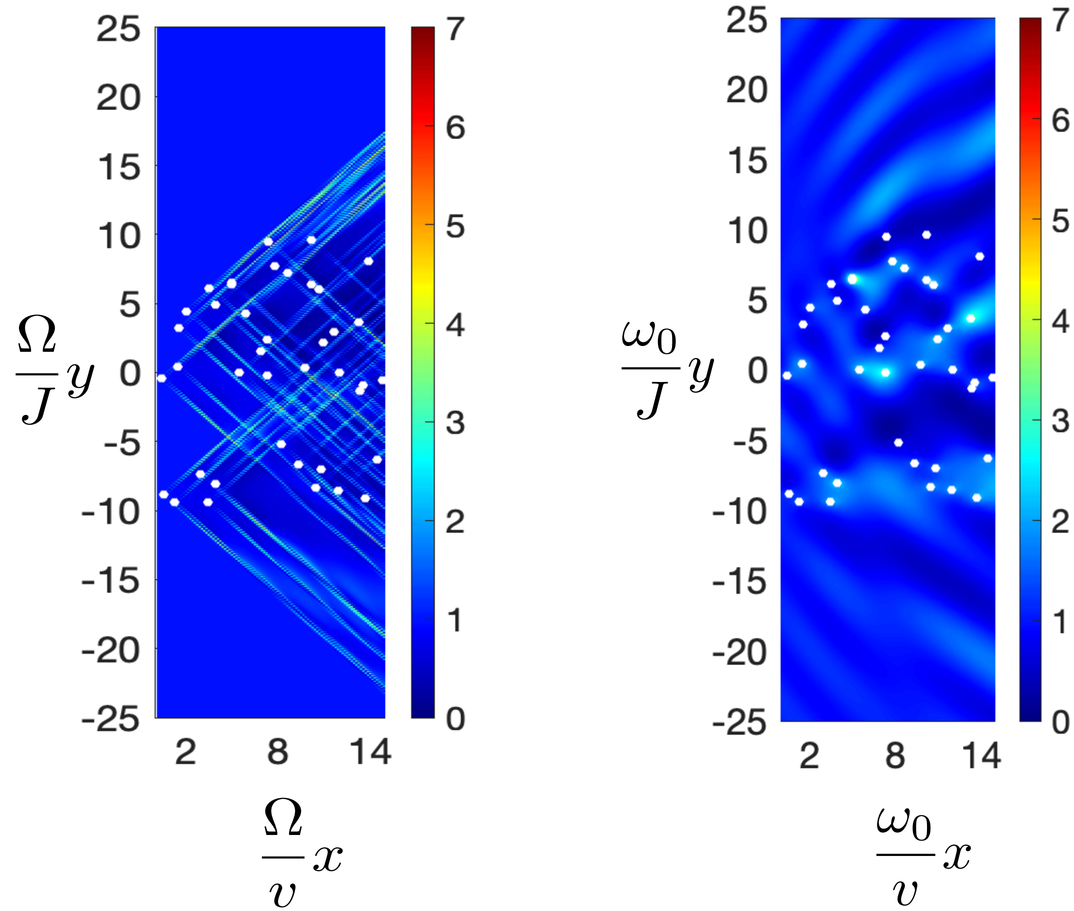
This is a (1+1) dimensional Dirac equation, where the coordinate x is timelike, y is spacelike and Ω plays the role of the mass. Here the nondimensionalized Dirac operator $D = i\alpha\partial_y + \beta$ with $D^2 = -\partial_y^2 + 1$.

For an antichiral array, the group velocities alternate in sign with $v_1 = -v_2 = v$ and $\Omega = 0$. It follows that ψ obeys

$$iv\beta\partial_x\psi + iJ\alpha\partial_y\psi + V(x,y)\psi = E\psi \ .$$

This is a (2+0) dimensional Dirac equation, where both the x and y coordinates are spacelike. Here the nondimensionalized Dirac operator $D = i\beta\partial_x + i\alpha\partial_y$ with $D^2 = -\Delta$.

Chiral and antichiral arrays



Plots of the total probability density $|\psi_1|^2 + |\psi_2|^2$.

Inverse problems

Consider a time-harmonic solution of

$$\begin{aligned} i\partial_t \psi &= c(-\Delta)^{1/2}\psi + g\rho(\mathbf{x})a \ , \\ i\partial_t a &= g\psi + \Omega a \ . \end{aligned}$$

This leads to the boundary value problem

$$\begin{aligned} (-\Delta)^{1/2}u + \eta(\mathbf{x})u = 0 \quad &\text{in} \quad \Omega \ , \\ u = f \quad &\text{in} \quad \Omega^c \ , \end{aligned}$$

where the potential $\eta(\mathbf{x}) = g^2/(\omega - \Omega)\rho(\mathbf{x})$.

We define $\Lambda_\eta(f) = (-\Delta)^{1/2}u\Big|_{\Omega^c}$. The inverse problem is to recover η from Λ_η. Due to nonlocality, the problem is formally determined with a single source. By unique continuation (Ghosh, Ruland and Uhlmann), there is an inversion formula

$$\eta(\mathbf{x}) = -\frac{(-\Delta)^{1/2}u(\mathbf{x})}{u(\mathbf{x})} \ ,$$

with measurements taken on an open subset of Ω^c. The unique continuation is not stable.

Instead, consider the scattering problem (in dimension three)

$$(-\Delta)^{1/2}\psi + \eta(\mathbf{x})\psi = k\psi \ ,$$

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}| \left(\frac{\partial}{\partial|\mathbf{x}|}\psi_s - k\psi_s \right) = 0 \ .$$

Here the scattered field $\psi_s = \psi - \psi_i$, where the incident field ψ_i obeys $(-\Delta)^{1/2}\psi_i - k\psi_i = 0$.

In the far zone, the scattered field is of the form

$$\psi_s \sim \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\phi(\mathbf{k},\mathbf{k}') \ ,$$

where the scattering amplitude ϕ depends on the incoming and outgoing wavevectors $\mathbf{k}$ and $\mathbf{k}'$.

The inverse problem is to recover η from measurements of ϕ.

Scattering theory

The field obeys the Lippmann-Schwinger equation

$$\psi(\mathbf{x}) = \psi_i(\mathbf{x}) + \int G(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\eta(\mathbf{y})d\mathbf{y} \ .$$

By iterating the above we obtain the Born series

$$\psi(\mathbf{x}) = \psi_i(\mathbf{x}) + \int G(\mathbf{x},\mathbf{y})\eta(y)\psi_i(\mathbf{y})d\mathbf{y}$$
$$+ \int G(\mathbf{x},\mathbf{y})\eta(\mathbf{y})G(\mathbf{y},\mathbf{y}')\eta(\mathbf{y}')\psi_i(\mathbf{y}')d\mathbf{y}d\mathbf{y}' + \cdots \ .$$

The outgoing Green's function is given by

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{|\mathbf{q}| - k - i\epsilon}d\mathbf{q} \ .$$

Note the identity

$$G(\mathbf{x},0) = -\frac{1}{2\pi|\mathbf{x}|}\frac{\partial}{\partial|\mathbf{x}|}g(\mathbf{x}) \ ,$$

where

$$g(\mathbf{x}) = \frac{1}{2\pi} e^{ik|\mathbf{x}|} E_1(ik|\mathbf{x}|) + ie^{ik|\mathbf{x}|} \ ,$$

The exponential integral is defined by

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt$$

and

$$E_1(x) = -\gamma - \log x - \sum_{n=1}^{\infty} \frac{(-x)^n}{n\, n!}$$

Thus the near field decays as $1/|\mathbf{x}|^2$ and the far field decays as $1/|\mathbf{x}|$.

In the far field

$$G(\mathbf{x}, \mathbf{y}) = k \ \frac{e^{ik|\mathbf{x}|}}{2\pi|\mathbf{x}|} \ e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} \left[1 + O\left(\frac{1}{|\mathbf{x}|}\right)\right] \ .$$

We then see that the scattered field is given by

$$\psi_s \sim k \ \frac{e^{ik|\mathbf{x}|}}{2\pi|\mathbf{x}|} \int e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} \eta(\mathbf{y}) \psi(\mathbf{y}) d\mathbf{y} \ .$$

By removing the geometrical factor, the scattering amplitude is of the form

$$\phi = \int e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} \eta(\mathbf{y})\psi(\mathbf{y})d\mathbf{y} \ .$$

Taking the incident field $\psi_i(\mathbf{x}) = e^{i\mathbf{k}'\cdot\mathbf{x}}$ and using the Born series for ψ, we find that

$$\phi = K_1(\eta) + K_2(\eta,\eta) + K_3(\eta,\eta,\eta) + \cdots \ ,$$

$$K_n(\eta_1,\ldots,\eta_m)(\mathbf{k},\mathbf{k}') = \int e^{-i\mathbf{k}\cdot\mathbf{y}_1 - \mathbf{k}'\cdot\mathbf{y}_n} G(\mathbf{y}_1,\mathbf{y}_2)\ldots G(\mathbf{y}_{n-1},\mathbf{y}_n) \\ \times \eta_1(\mathbf{y}_1)\ldots\eta_m(\mathbf{y}_n)\, \mathrm{d}\mathbf{y}_1\ldots\mathrm{d}\mathbf{y}_n \ ,$$

and $\mathbf{k} = k\hat{\mathbf{x}}$ and $\mathbf{k}' = k\hat{\mathbf{x}}'$.

We can estimate the norm of K_m by

$$\|K_m\| \le \nu\mu^{m-1} \ ,$$

for suitable constants μ and ν. Here $K_m : L^2(B_a \times \cdots \times B_a) \to L^2(S^2)$, where η is supported in the ball B_a.

Inverse Born series

Let X and Y be Banach spaces and $K_m : X^m \to Y$ be a multilinear operator. Here X^m indicates the m-fold tensor product $X^m = X \otimes \cdots \otimes X$ equipped with the projective norm. Consider the operator $\mathcal{F} : X \to Y$ defined by

$$\mathcal{F}[\eta] = \sum_{m=1}^{\infty} K_m(\eta, \ldots, \eta).$$

The forward problem is to evaluate the map $\mathcal{F} : \eta \mapsto \phi$ for $\eta \in X$ and $\phi \in Y$. We refer to the above as the Born series.

The inverse problem is to determine η assuming ϕ is known. That is, we wish to construct a map $\mathcal{I} : Y \to X$ which is, in a suitable sense, the inverse of $\mathcal{F}$. We define the operator $\mathcal{I}$ by

$$\mathcal{I}[\phi] = \sum_{m=1}^{\infty} \mathcal{K}_m(\phi).$$

This is the inverse Born series.

To find the operators $\mathcal{K}_m : Y^m \to X$, substitute the series for η into the series for ϕ and equate terms of the same order in ϕ. We find that $\mathcal{K}_m$ is homogeneous of degree m and is given by

$$\begin{aligned}
&\mathcal{K}_1 K_1 = I, \\
&\mathcal{K}_2(\phi) = -\mathcal{K}_1\left(K_2(\mathcal{K}_1(\phi), \mathcal{K}_1(\phi))\right), \\
&\mathcal{K}_m(\phi) = -\sum_{n=2}^{m} \sum_{i_1+\cdots+i_n=m} \mathcal{K}_1 K_n\left(\mathcal{K}_{i_1}(\phi), \ldots, \mathcal{K}_{i_n}(\phi)\right).
\end{aligned}$$

We note that inversion of K_1 is simple since

$$K_1(\eta)(\mathbf{k}, \mathbf{k}') = \int e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{y}} \eta(\mathbf{y}) d\mathbf{y}$$

with $|\mathbf{k}| = |\mathbf{k}'| = k$ yields a band-limited Fourier transform. Thus the highest frequency present in the reconstruction is $2k$.

The following theorem establishes a sufficient condition for convergence of the inverse Born series. Let $B_{R,X}$ denote the ball of radius R centered at the origin in the Banach space X

Theorem (Hoskins and S)**.** Let μ and ν be positive constants. Suppose that $\|K_m\| \leq \nu\mu^{m-1}$ for $m = 1, 2, \cdots$. The inverse Born series converges if $\|\mathcal{K}_1\phi\|_X < r$, where the radius of convergence r is given by

$$r = \frac{1}{2\mu}\left[\sqrt{16C^2+1} - 4C\right],$$

where $C = \max\{2, \|\mathcal{K}_1\|\nu\}$. Moreover, if $\mathcal{K}_1\phi \in B_r(X)$ then the inverse operator $\mathcal{I}$ maps $B_r(X)$ into $B_{r_0}(Y)$, with $r_0 = 2\mu/\sqrt{16C^2+1}$.

We characterize the approximation error as follows.

Theorem (Hoskins and S)**.** Suppose that the previous hypotheses hold and that the Born and inverse Born series converge. Let $\tilde{\eta}$ denote the sum of the inverse Born series and $\eta_1 = \mathcal{K}_1\phi$. Setting $\mathcal{M} = \max\{\|\eta\|_X, \|\tilde{\eta}\|_X\}$, we further assume that

$$\mathcal{M} < \frac{1}{\mu}\left(1 - \sqrt{\frac{\nu\|\mathcal{K}_1\|}{1+\nu\|\mathcal{K}_1\|}}\right).$$

Then the approximation error can be estimated as follows:

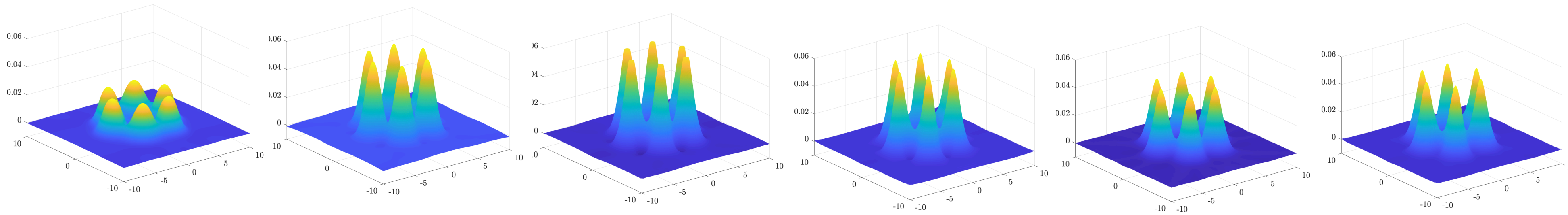
$$\left\|\eta - \sum_{m=1}^{N} \mathcal{K}_m(\phi)\right\|_X \leq M \left(\frac{\|\eta_1\|_X}{r}\right)^{N+1} \frac{1}{1-\frac{\|\eta_1\|_X}{r}} + \left(1 - \frac{\nu\|\mathcal{K}_1\|}{(1-\mu\mathcal{M})^2} + \nu\|\mathcal{K}_1\|\right)^{-1} \|(I - \mathcal{K}_1 K_1)\eta\|_X ,$$

where $M = \frac{2\mu}{\sqrt{16C^2+1}}$.

Numerical reconstructions

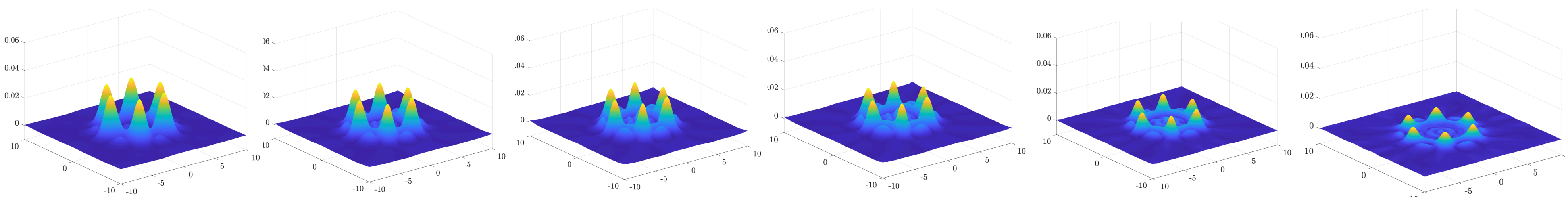
Simulations were performed in 2D with $k = 2\pi$, 100 sources and 100 detectors, and in 3D with 1000 sources and 1000 detectors.

Two dimensions



1 term 2 terms 3 terms 4 terms 5 terms 6 terms

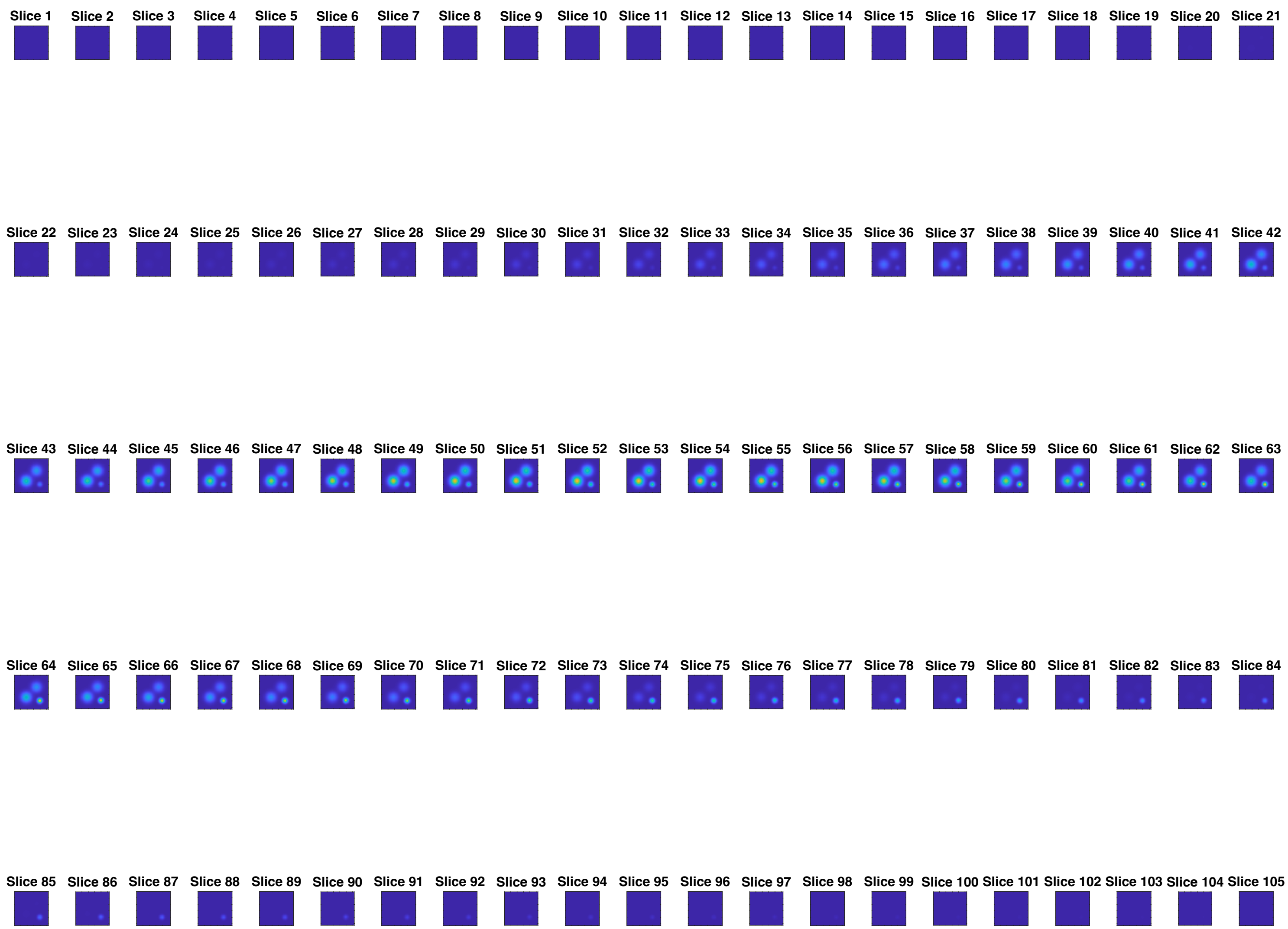
Reconstructions



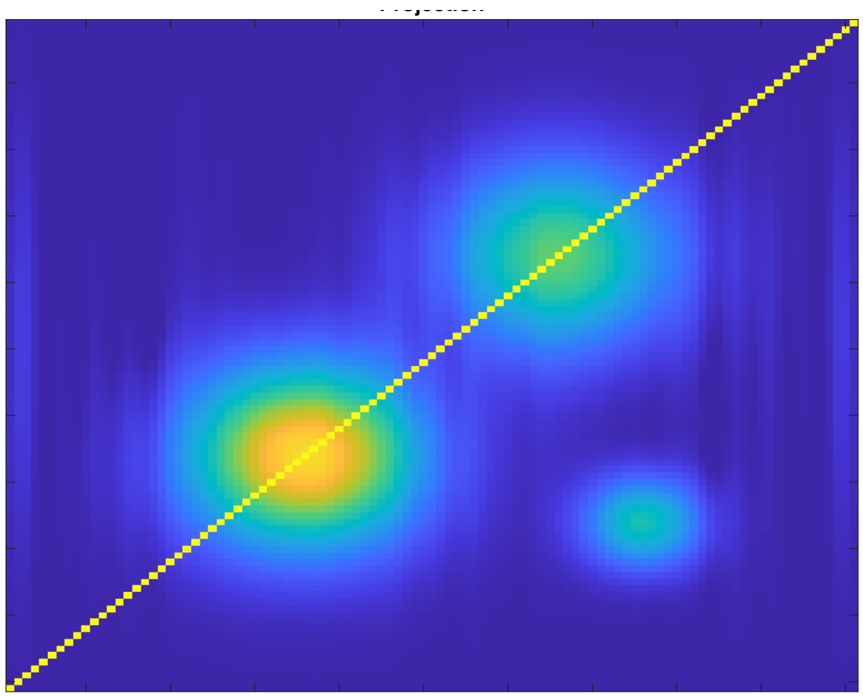
1 term 2 terms 3 terms 4 terms 5 terms 6 terms

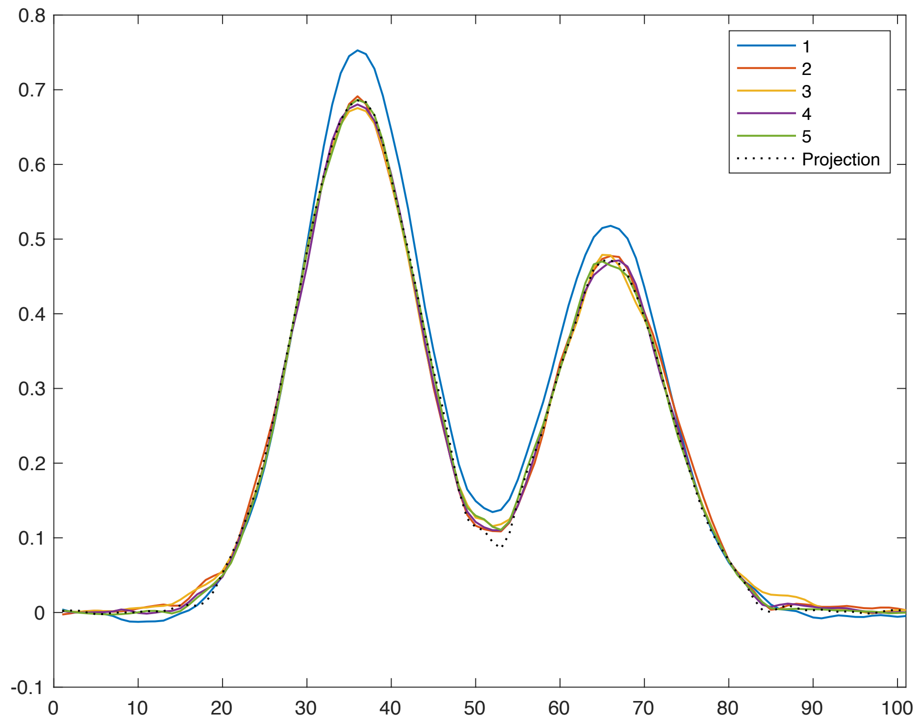
Error

Three dimensions



Slice 50





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The End