Mathematics of in-gap interface modes in photonic/phononic structures

Hai Zhang

Department of Mathematics, HKUST

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Plan of the two lectures

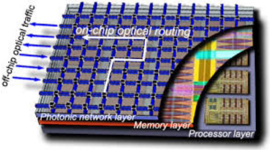
We plan to go through the following topics:

1. Introduction;
2. Interface eigenvalue bifurcated from a Dirac point in one-dimensional structure;
3. Bulk-interface correspondences for one-dimensional topological materials with inversion symmetry;
4. In-gap interface eigenvalue in a waveguide bifurcated from a Dirac point;
5. In-gap interface eigenvalue in a waveguide bifurcated from a Dirac point without band gap opening;
6. Integer Quantum Hall Effect in square lattices photonic structure;
7. Interface modes in honeycomb photonic structure.

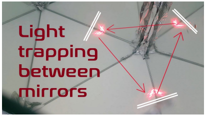
Can we trap light?

In 1862, Maxwell concluded that light was a form of electromagnetic radiation. In 1873, he presented a full mathematical description of the behavior of electromagnetic fields, using Maxwell's equations. A natural question is: can we control light or simply trap light? While the question is of interest in fundamental physics, there are many applications if we do can trap light:

1. Communication: optical fibers;
2. Information processing: photonic integrated circuits;
3. Green energy: more efficient solar cells.

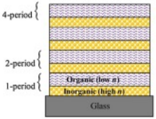


A naive idea to trap light: reflection mirrors

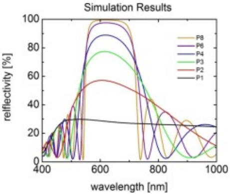


But mirrors made of metals also absorb light, we want the material to be lossless.

Solution: Bragg mirror (a periodic multi-layer dielectric stacks), first studied by L. Rayleigh in 1887.



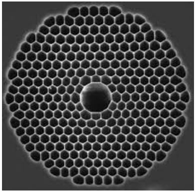
88% reflectivity for a 4-period structure.

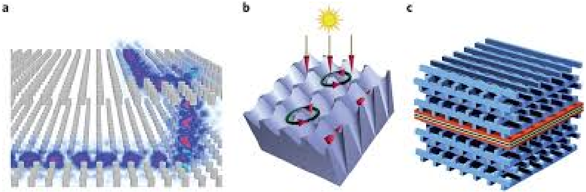


Bragg mirror is a 1-D PC (photonic crystal), for trapping light in 3D, we need 3D PCs + some new ideas.

Key ideas for trapping of light: band gaped PCs with defects

1. Anderson (1958): electrons could be trapped in a disordered material where the atoms are arrayed randomly.
2. S. John(1987): “Strong localization of photons in certain disordered dielectric superlattices”.
3. E. Yablonovitch(1987): If a 3D PC has an band gap which overlaps the electronic band edge, then spontaneous emission can be rigorously forbidden.
4. First realization of trapping of light (1997): D. Wiersma, A.Lagendijk, using a powder of gallium arsenide.





Mathematical work on localization of light in lossless medium

The ideas and proposals of physicists were given a rigorous foundation due to the work of A. Figotin and A. Klein who showed that trapping is possible by using PCs with band gaps for classic waves in 3D:

1. Existence of localized “defect eigenmode” provided that the periodic medium is perturbed by a single sufficiently large defect, 1997, 1998.
2. Existence of infinite number of localized eigenmodes with frequencies dense in an interval contained in the spectral bandgap, provided that the periodic medium is perturbed by a random array of defects with certain natural conditions, 1996, 1997.

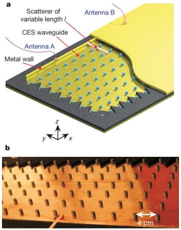
Remark: defect eigenmode is the form of the waves that are trapped near the defect.

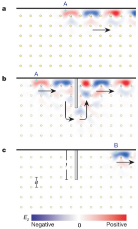
New way for localization: topological photonics

- Topological insulators refer to electronic materials that are insulating in their bulk, but are conductive on their boundaries or interfaces. The research field develops rapidly since the experimental discovery of integer quantum Hall effect by K. Klitzing in 1980. Subsequently, Thouless et al (1982) and Kohmoto (1985) relates to integer in the quantized Hall conductance to a topological invariant of the system, the Chern number.

- Electromagnetic analogues of quantum-Hall-effect in photonic crystals: Raghu-Haldane, '08.

- Experimental realization in 2D magneto-optical photonic crystal in the microwave regime: Wang-Chong-Joannopoulos-Soljacic, '09.



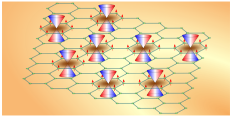


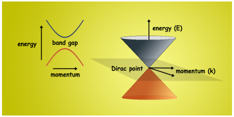
Mathematical studies in topological materials

• Interface/edge modes can be obtained using two approaches: bifurcation of spectral degenerate points and bulk-interface/edge correspondence.

• A class of widely studied degenerate points are Dirac points: linear degenerate points where topological phase transition takes place.

Mathematical theory: Ablowitz-Zhu, '12, Fefferman-Weinstein, '12, Ammari-Hiltunen-Yu, '20, Lin-Li-Zhang, '2023$\cdots$.





• The bulk-edge correspondence: The existence of interface/edge modes supported at the interface of two structures with distinct topological invariants, and the characterization of the number of modes in terms of bulk indices.

• Bulk-edge correspondence in discrete models: transfer matrix and Riemann surfaces (Hatsugai '93), K-theory (Prodan-Schulz-Baldes '04, '16), Elbau-Graf '02, Elgart-Graf-Schenker '05, Mong and Shivamoggi '11, Graf-Port '13, Taarabt '14, , etc.

Mathematical studies of topological materials in continuous models

1. Integer quantum Hall system: Kellendonk, Schulz-Baldes, 2004; Combes, Germinet, 2005, etc.
2. Domain wall models of various 1-D and 2-D Schrödinger equations and others: Fefferman, Weinstein, Drout, Lee, Thorp, 2017-;
3. Dirac operators: G. Bal, 2019-2021; Drout, 2022-
4. Topological metamaterials made of high contrast bubbles: Ammari, Davies, Hiltunen, Yu, et.al., 2018-;
5. . . .

Interface eigenvalue bifurcated from a Dirac point in one-dimensional structure

Reference: J. Lin and H. Zhang, *Mathematical theory for topological photonic materials in one dimension*, Journal of Physics A, 2022.

Topologically protected states[1] in 1D quantum system [2]

$$\begin{cases} \mathscr{H}_\delta \Psi_\delta = \lambda \Psi_\delta \quad \Psi_\delta \in L^2(\mathbb{R}), \\ \mathscr{H}_\delta \equiv -\partial_x^2 + V_e(x) + \delta \cdot \kappa(\delta \cdot x) W_o(x). \end{cases}$$

Domain wall model:

1. The unperturbed operator $\mathscr{H}_{\delta=0}$ has a Dirac point at $(E = E_*, p = \pi)$.
2. $V_e \in C^\infty(\mathbb{R}), V_e = \sum_{k \in 2\mathbb{Z}_+} v_k cos(2\pi kx)(v_k \in \mathbb{R})$
3. $W_o \in C^\infty(\mathbb{R}), W_o = \sum_{k \in 2\mathbb{Z}_+ +1} w_k cos(2\pi kx)(w_k \in \mathbb{R}))$.
4. The perturbed Hamiltonians are different for $-\delta W_0$ and $+\delta W_0$.
5. $\kappa(\pm\infty) = \pm 1$: two different bulk media at $\pm\infty$ are connected adiabatically by an interface layer of size $1/\delta$.

Using two-scale expansion method, the existence and asymptotoics of the interface eigenvalues can be obtained.

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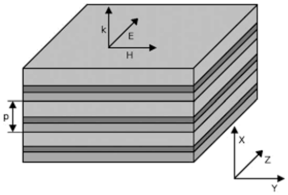
[1] These states are interface modes which are confined near the interface between two media.

[2] C. Fefferman, J. Lee-Thorp, and M. Weinstein, *Topologically protected states in one-dimensional systems*, **247**, Memoirs of American Mathematical Society, 2017

Our problem setup



Consider the 1D photonic structure modeled by

$$\mathscr{L}\psi = -\frac{1}{\varepsilon(x)}\frac{d}{dx}\left(\frac{1}{\mu(x)}\frac{d\psi}{dx}\right) \quad \text{for } x \in \mathbb{R},$$

where the permittivity $\varepsilon(x)$ and the permeability $\mu(x)$ are piecewisely continuous functions with the period 1:

$$\varepsilon(x) = \varepsilon(x+1), \quad \mu(x) = \mu(x+1).$$

Assume that both $\varepsilon(x)$ and $\mu(x)$ are positive-valued so that the photonic/phononic system is time-reversal symmetric.

Band structure of the spectrum of $\mathscr{L}$ - Floquet-Bloch theory

• The spectrum of $\mathscr{L}$ can be obtained by solving a family of eigenvalue problems indexed by the quasi-momentum $k \in [-\pi, \pi]$:

$$\mathscr{L}\psi(x) = E\psi(x) \quad \text{in } L^2_k = \{\psi \in L^2_{loc} : \psi(x+1) = e^{ik}\psi(x)\}.$$

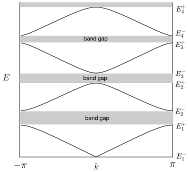
Each eigenvalue problem is self-adjoint and attains a discrete set of real eigenvalues

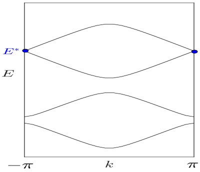
$$0 \le E_1(k) \le E_2(k) \le \cdots \le E_j(k) \le \cdots.$$

• $\sigma(\mathscr{L}) = \bigcup_{j\ge 1} [E_j^-, E_j^+]$, where

$$E_1^- < E_1^+ \le E_2^- < E_2^+ \le E_3^- < E_3^+ \cdots$$

• A Dirac point occurs when two spectral bands intersect linearly at the intersection point.





Idea: open a band gap near a Dirac point and create a point spectrum inside

The transfer matrix and Dirac point

- For each $E \in \mathbb{R}$, the transfer matrix

$$\Psi_E(x) = (\Psi_{E,1}(x), \Psi_{E,2}(x)) = \begin{pmatrix} \psi_{E,1}(x) & \psi_{E,2}(x) \\ \frac{1}{\mu(x)}\psi'_{E,1}(x) & \frac{1}{\mu(x)}\psi'_{E,2}(x) \end{pmatrix},$$

where

$$\begin{cases} (\mathscr{L} - E)\psi_{E,1} = 0, & \psi_{E,1}(0) = 1, \ \frac{1}{\mu(0)}\psi'_{E,1}(0) = 0, \\ (\mathscr{L} - E)\psi_{E,2} = 0, & \psi_{E,2}(0) = 0, \ \frac{1}{\mu(0)}\psi'_{E,2}(0) = 1. \end{cases}$$

- The solution of $\mathscr{L}\psi(x) = E\psi(x)$ can be expressed by

$$\begin{pmatrix} \psi(x) \\ \frac{1}{\mu(x)}\psi'(x) \end{pmatrix} = \Psi_E(x) \begin{pmatrix} \psi(0) \\ \frac{1}{\mu(0)}\psi'(0). \end{pmatrix}$$

- Let $M(E) = \Psi_E(1)$, $D(E) = Tr\ M(E)$. Fact: $det\ M(E) = 1$.

Theorem

Dirac points can only occur at $k^* = 0$ or $k^* = \pi$ with $D(E^*) = \pm 2$ and $D'(E^*) = 0$. Furthermore, $(k^* = 0, E^*)$ is a Dirac point if $D'(E^*) = 0$, $D(E^*) = 2$. $(k^* = \pi, E^*)$ is a Dirac point if $D'(E^*) = 0$, $D(E^*) = -2$.

Perturbation of Dirac point

- Assume that the operator $\mathscr{L}$ attains a Dirac point (k^*, E^*) with $k^* = 0$:

$$E^* = E_j^+ = E_{j+1}^- = E_j(0) = E_{j+1}(0).$$

- Perturb the operator $\mathscr{L}$ with

$$\begin{cases} \mu(x) \to \mu(x) + \delta\tilde{\mu}(x), \\ \varepsilon(x) \to \varepsilon(x) + \delta\tilde{\varepsilon}(x), \end{cases}$$

where $|\delta| \ll 1$, and $\tilde{\mu}(x)$ and $\tilde{\varepsilon}(x)$ are periodic functions with $\|\tilde{\mu}\|_{L^\infty} + \|\tilde{\varepsilon}\|_{L^\infty} = 1$.

- The perturbed operator:

$$\mathscr{L}_\delta \psi(x) = -\frac{1}{\varepsilon(x) + \delta\tilde{\varepsilon}(x)} \left(\frac{1}{\mu(x) + \delta\tilde{\mu}(x)} \psi'(x) \right)'.$$

Band gap opening at the Dirac point due to perturbation

$$a_1 = \frac{\partial^2 D}{\partial E^2}(E^*), \quad a_2 = \frac{\partial^2 D}{\partial E \partial \delta}(E^*), \quad a_3 = \frac{\partial^2 D}{\partial \delta^2}(E^*).$$

Theorem

If $a_2^2 - a_1 a_3 > 0$, then for $\delta > 0$ sufficiently small, there exists a band gap between the j-th and the $(j+1)$-th band for the perturbed operator $\mathscr{L}_\delta$.
If $a_3 > 0$, then there exists a common band gap between the j-th and the $(j+1)$-th band for the operator $\mathscr{L}_\delta$ and $\mathscr{L}_{-\delta}$.

Remark: It can be proven that $a_1 < 0$, which is independent of the perturbation.

Existence of interface mode

- Let

$$\mathscr{L}_{\delta,\pm}\psi(x) = -\frac{1}{\varepsilon_{\delta,\pm}(x)}\frac{d}{dx}\left(\frac{1}{\mu_{\delta,\pm}(x)}\frac{d\psi}{dx}\right),$$

where $\varepsilon_{\delta,\pm}(x) = \varepsilon(x) \pm \delta\tilde{\varepsilon}(x)$ and $\mu_{\delta,\pm}(x) = \mu(x) \pm \delta\tilde{\mu}(x)$.

- Define

$$\tilde{\mathscr{L}}_{\delta} = \begin{cases} \mathscr{L}_{\delta,-}, & x < 0; \\ \mathscr{L}_{\delta,+}, & x > 0. \end{cases}$$

Theorem

Assume that the operator $\mathscr{L}$ attains a Dirac point $(k^* = 0$ or π, $E^*)$ at the intersection of the j-th and $(j+1)$-th band. Further assume that $a_3 > 0$. There exists an interface mode for the operator $\tilde{\mathscr{L}}_{\delta}$ for $\delta \ll 1$.

Remark. The assumptions hold if μ and ε are even functions, and $\tilde{\mu}$ and $\tilde{\varepsilon}$ are odd functions.

Key ingredient in the proof: Impedance function in the band gap

For E in the band gap, all L^2-solutions to the equation $(\mathscr{L}-E)\psi=0$ over the left half-line $(-\infty,0]$ span a one-dimensional space. Let $\psi_{L,E}$ be one of these solutions. We define the left **impedance function** for the operator $\mathscr{L}$ in the left half-line to be

$$\xi_L(E) := \frac{\psi_{L,E}(0)}{\frac{1}{\mu(0)}\psi'_{L,E}(0)}, \quad \text{if } \psi'_{L,E}(0)\neq 0.$$

In the case where $\psi'_{L,E}(0)=0$, we set formally $\xi_L(E)=\infty$. In a similar way, we define the right impedance function $\xi_R(E)$ for the right half-line.

Lemma

Assume that E lies in a common spectral band gap of $\mathscr{L}_{\delta,-}$ and $\mathscr{L}_{\delta,+}$, then there exists an interface mode at energy level E for the operator $\tilde{\mathscr{L}}_\delta$ if and only if

$$\xi_{L,-}(E)=\xi_{R,+}(E).$$

Key steps: asymptotic expansion of the "Bloch modes" in the band gap, asymptotic expansion of the impedance functions.

Bulk-interface correspondences for one-dimensional topological materials with inversion symmetry

Reference I: G.C. Thiang and H. Zhang, *Bulk-interface correspondences for one-dimensional topological materials with inversion symmetry*, Proceedings of Royal Society A, 2023.
Reference II: J. Lin and H. Zhang, *Mathematical theory for topological photonic materials in one dimension*, Journal of Physics A, 2022.

Problem setup

The periodic differential operator of interest is given by

$$\mathscr{L}\psi = -\frac{1}{\varepsilon(x)}\frac{d}{dx}\left(\frac{1}{\mu(x)}\frac{d\psi}{dx}\right) \quad \text{for } x \in \mathbf{R}, \tag{1}$$

and the coefficients ε, μ satisfy the following two conditions:

- The permittivity $\varepsilon(x)$ and the permeability $\mu(x)$ are piecewise continuous positive-valued functions with period one:

$$\varepsilon(x) = \varepsilon(x+1), \quad \mu(x) = \mu(x+1).$$

- $\mathscr{P}\mathscr{L} = \mathscr{L}\mathscr{P}$, where $\mathscr{P}$ is the **parity operator** defined by

$$\mathscr{P}\psi(x) = \psi(-x).$$

Under the above assumptions, $\varepsilon(x) = \varepsilon(-x)$, $\mu(x) = \mu(-x)$, or equivalently, $\varepsilon(x) = \varepsilon(1-x)$, $\mu(x) = \mu(1-x)$. Also, $\mathscr{L}$ is **time-reversal symmetric** in the sense that it commutes with the operation of complex conjugation.
Such systems were investigated by M. Xiao, Z.Q. Zhang, C.T. Chan, Surface impedance and bulk band geometric phases in one-dimensional systems, Phys. Rev. X, 2014.

Change of parity for the Bloch modes at band gap edges.

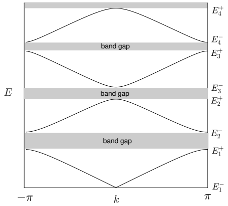
We say that a Bloch mode $\varphi_{j,k}$ has even-parity (odd-parity) if $\varphi_{j,k}$ is an even (odd) function.

Lemma

The Bloch modes $\varphi_{j,k}$ are even or odd when $k = 0$ or π at an isolated band.

Proposition

Let $\mathscr{L}$ be a periodic operator of the form (1). Assume that there is a band gap between the j-th and $(j+1)$-th bands. Then the Bloch modes at (k^*, E_j^+) and at (k^*, E_{j+1}^-) have different parities, where $k^* = 0$ or π.



Bulk topological index for inversion symmetric system

Assume that there is a gap between the j-th and $(j+1)$-th spectral bands of $\mathscr{L}$. Recall that the lower edge of the band gap, i.e. the maximum of the j-th band, E_j^+, is achieved at either $k=0$ or π. With respect to this band gap, we define the following bulk index:

$$\gamma_j =: \text{ the parity of the Bloch mode at } E_j^+.$$

We can show that

$$\gamma_j = (-1)^{j-1} e^{i\sum_{m=1}^{j} \theta_m},$$

where θ_m is the Zak phase for the m-th isolated band.

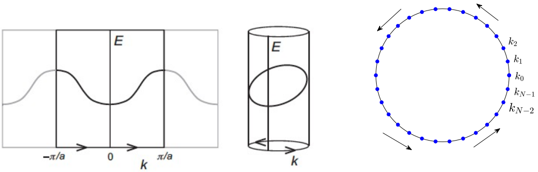
Review on the Zak phase

- The normalized Bloch mode $\varphi_{j,k}$ (the eigenfunction corresponding to the eigenvalue $E_j(k)$) can be written as $\varphi_{j,k}(x) = e^{ikx}u_{j,k}(x)$, where $u_{j,k}(x)$ is a periodic function satisfying $u_{j,k}(x) = u_{j,k}(x+1)$.

- The discrete Zak phase over the j-th band:

$$\theta_j^{(N)} = \sum_{n=1}^{N-1} -\mathrm{Im}\ln\left(u_{j,k_{n+1}}, u_{j,k_n}\right)_X - \mathrm{Im}\ln\left(e^{-i2\pi x}u_{j,k_0}, u_{j,k_{N-1}}\right)_X \; mod\ 2\pi,$$

where $k_n = -\pi + \frac{2\pi n}{N}$. The last term comes from the observation that $u_{j,\pi}(x) = e^{-i2\pi x}u_{j,-\pi}(x)$.



- The continuous Zak phase over the j-th band (assuming that $\varphi_{j,-\pi} = \varphi_{j,\pi}$):

$$\theta_j = i\int_{-\pi}^{\pi}\left(\frac{\partial u_{j,k}}{\partial k}, u_{j,k}\right)_X dk \; mod\ 2\pi.$$

Bulk-interface correspondence for the inversion symmetric systems

- Let

$$\mathscr{L}_j\psi = -\frac{1}{\varepsilon_j(x)}\frac{d}{dx}\left(\frac{1}{\mu_j(x)}\frac{d\psi}{dx}\right), \quad j=1,2,$$

where $\varepsilon_j(x)=\varepsilon_j(1-x)$, $\mu_j(x)=\mu_j(1-x)$.

- The differential operator for the joint structure is given by

$$\mathscr{A}\psi(x) = \begin{cases} \mathscr{L}_1\psi(x), & x<0, \\ \mathscr{L}_2\psi(x), & x>0. \end{cases}$$

Theorem

Assume that the following holds:

(i) *The operators $\mathscr{L}_1$ and $\mathscr{L}_2$ attain a common band gap*

$$I := (E_{m_1}^{(1),+}, E_{m_1+1}^{(1),-}) \cap (E_{m_2}^{(2),+}, E_{m_2+1}^{2,-}) \neq \emptyset$$

for certain positive integers m_1 and m_2.

(ii) *The bulk topological indices $\gamma_{m_1}^{(1)} \neq \gamma_{m_2}^{(2)}$ for $\mathscr{L}_1$ and $\mathscr{L}_2$,*

Then there exists a unique interface mode $\psi \in L^2(\mathbb{R})$ for the operator $\mathscr{A}$.

Key observation of the impedance functions in the band gap

Lemma

Assume that there is a band gap between the j-th and the $(j+1)$-th bands. Then the following hold for $E \in (E_j^+, E_{j+1}^-)$:

1. If the Bloch mode at the band edge (k, E_j^+) has odd-parity for $k = 0$ or π, then $\xi_R(E)$ is strictly decreasing, with $\xi_R(E) \to 0$ as $E \to E_j^+$ and $\xi_R(E) \to -\infty$ as $E \to E_{j+1}^-$; On the other hand, $\xi_L(E)$ is strictly increasing, with $\xi_L(E) \to 0$ as $E \to E_j^+$ and $\xi_L(E) \to +\infty$ as $E \to E_{j+1}^-$.
2. Similar conclusions if the Bloch mode at (k, E_j^+) has even-parity.

A consequence on the dislocation model

Proposition

Let

$$\mathscr{L}_1 = -\frac{1}{\varepsilon_1(x)}\frac{d}{dx}\left(\frac{1}{\mu_1(x)}\frac{d}{dx}\right).$$

Assume that the spectrum of $\mathscr{L}_1$ has a band gap between the j-th and $(j+1)$ band. Further assume that the maximum of the j-th spectral band and the minimum of the $(j+1)$-th spectral band are achieved at $k=\pi$. Let $\mathscr{L}_2$ be the $1/2$ shifted version of $\mathscr{L}_1$, in the sense that the corresponding coefficients ε_2 and μ_2 are related to those of $\mathscr{L}_1$ by

$$\varepsilon_2(x) = \varepsilon_1(x-1/2); \quad \mu_2(x) = \mu_1(x-1/2).$$

Then there exists a unique interface mode in the band gap between the j-th and $(j+1)$ band for the glued operator $\tilde{\mathscr{L}}$.

Remark: a shift of origin by $1/2$ will change Zak phase by π.

Extension to other one-dimensional systems

Remark

All the results can be extended to electronic systems modeled by Schrödinger operators as below:

$$\mathscr{L}_j = -\frac{d^2}{dx^2} + V_j, \quad j = 1, 2,$$

where V_j are real-valued piecewise continuous functions in one dimension that are periodic and are even.

Stability of interface modes

- For a photonic system with an interface mode in a common spectral band gap of two operators $\mathscr{L}_1$ and $\mathscr{L}_2$, we perturb it with a defect region (d_1, d_2), and the relative permittivity and permeability are

$$\varepsilon(x) = \begin{cases} \varepsilon_1(x-d_1), & x < d_1, \\ \varepsilon_d(x), & d_1 < x < d_2, \\ \varepsilon_2(x-d_2), & x > d_2. \end{cases} \quad \text{and} \quad \mu(x) = \begin{cases} \mu_1(x-d_1), & x < d_1, \\ \mu_d(x), & d_1 < x < d_2, \\ \mu_2(x-d_2), & x > d_2. \end{cases}$$

Theorem

Assume that $\mathscr{L}_1$ and $\mathscr{L}_2$ attain a common band gap $I := (E_{m_1}^{(1),+}, E_{m_1}^{(1),-}) = (E_{m_2}^{(2),+}, E_{m_2+1}^{2,-})$ and the bulk topological indices $\gamma_{m_1}^{(1)}$ and $\gamma_{m_2}^{(2)}$ are different for the two operators. If

$$\max\left\{\|\mu\|_{L^\infty(d_1,d_2)}, E\|\varepsilon\|_{L^\infty(d_1,d_2)}\right\} \cdot (d_2 - d_1) < \frac{\pi}{2}$$

holds for any $E \in I$, then the perturbed operator attains an interface mode.

Theorem

Assume that $\mathscr{L}_1$ and $\mathscr{L}_2$ attain a common band gap and the bulk topological indices are different for the two operators. If $\varepsilon_d(x) \equiv \varepsilon_0$ and $\mu_d(x) \equiv \mu_0$ for certain constants ε_0 and μ_0, then the perturbed operator attains a localized state for any $\varepsilon_0 \geq 1$, $\mu_0 \geq 1$, and $d := d_2 - d_1 \geq 0$.

A quick summary so far

Present a mathematical theory for in-gap interface eigenvalue in 1D topological photonic/phononic structure:

1. Characterization of Dirac points;
2. Existence of in-gap interface eigenvalue bifurcated from a Dirac point;
3. Bulk-interface correspondence for systems with inversion symmetry;
4. Stability of interface modes.

In-gap interface eigenvalue in a waveguide bifurcated from a Dirac point

Reference: J. Y. Qiu, J. Lin, P. Xie and H. Zhang, Mathematical theory for the interface mode in a waveguide bifurcated from a Dirac point, preprint.

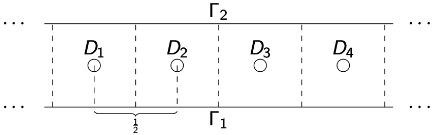
Related work:

H. Ammari, B. Davies, EO. Hiltunen, S.Yu, Topologically protected edge modes in one-dimensional chains of subwavelength resonators, 2020.

H. Ammari, B. Davies, EO. Hiltunen, Robust edge modes in dislocated systems of subwavelength resonators, 2022.

Problem setup: the unperturbed waveguide

We aim to create an in-gap interface eigenvalue in a waveguide by perturbing the following periodic structure with periodic one that has Dirac point.



Remark: There are two identical particles in a unit cell; Wave propagation can be modelled by

$$
\begin{cases}
(\Delta_x + \lambda) u(x;\lambda) = 0, \quad x \in \Omega, \\
u(x;\lambda) = 0, \quad x \in \bigcup_{n\in\mathbb{Z}} \partial D_n, \\
\dfrac{\partial}{\partial x_2} u(x;\lambda) = 0, \quad x \in \Gamma_1 \bigcup \Gamma_2.
\end{cases}
$$

Assumptions on the unperturbed structure

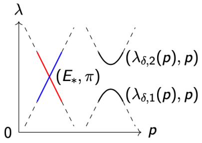
1. There is a Dirac point at $\lambda = \lambda_*, p = p_* = \pi$ which is at the intersection of the lowest two dispersion curves (due to the shift by $1/2$ sub-lattice symmetry).
2. λ_* is not an embedded eigenvalue.
3. $\lambda_* \neq (2m+1)\pi$ $(m \in \mathbb{Z})$.
4. Reflection symmetry. $\mathscr{P}\Omega = \Omega$ where $\mathscr{P}$: $\mathscr{P}(x_1, x_2) = (-x_1, x_2)$.
5. There exists analytic functions (in p) $(\mu_n(p), v_n(\cdot;p))$, $n = 1, 2$ such that

$$\mu_1(p) = \begin{cases} \lambda_1(p), & p \in [0,\pi), \\ \lambda_2(p), & p \in [\pi, 2\pi], \end{cases} \quad v_1(\cdot, p) = \begin{cases} u_1(\cdot;p), & p \in [0,\pi), \\ u_2(\cdot;p), & p \in [\pi, 2\pi], \end{cases}$$

and

$$\mu_2(p) = \begin{cases} \lambda_2(p), & p \in [0,\pi), \\ \lambda_1(p), & p \in [\pi, 2\pi], \end{cases} \quad v_2(\cdot, p) = \begin{cases} u_2(\cdot;p), & p \in [0,\pi), \\ u_1(\cdot;p), & p \in [\pi, 2\pi], \end{cases}$$

Moreover, $\mu_1'(p_*) = -\mu_2'(p_*) = \alpha_* \neq 0$.



The perturbed structures

$$\begin{cases} (\Delta_x + \lambda) u(x;\lambda) = 0, x \in \Omega_\delta, \\ u(x;\lambda) = 0, x \in \partial D_{n,\delta}, \\ \dfrac{\partial}{\partial x_2} u(x;\lambda) = 0, x \in \Gamma_1 \bigcup \Gamma_2. \end{cases} \Rightarrow \begin{cases} \mathscr{L}_\delta : H_b^2(\Omega_\delta) \subset L^2(\Omega_\delta) \to L^2(\Omega_\delta), \\ \qquad u \mapsto -\Delta u. \end{cases}$$

The glued operator:

$$\tilde{\mathscr{L}}_\delta \psi(x) := \begin{cases} \mathscr{L}_\delta \psi(x), & x_1 > 0, \\ \mathscr{L}_{-\delta} \psi(x), & x_1 < 0. \end{cases}$$

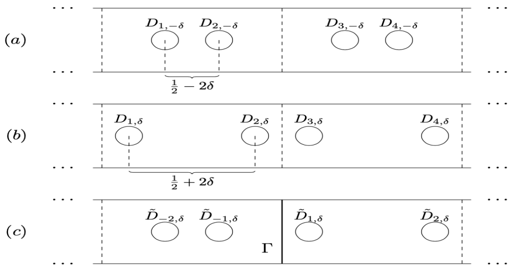


Figure: Waveguide with perturbation:(a) $\mathscr{L}_{-\delta}$, (b) $\mathscr{L}_{-\delta}$, (c) $\tilde{\mathscr{L}}_\delta$.

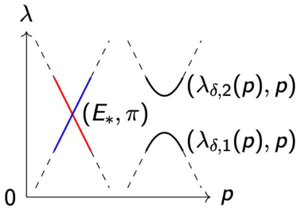
Asymptotic expansion and band gap opening

Proposition

Assume that $t_* \neq 0$ (t_* is a constant determined by the perturbation). Then the first two branches of dispersion relation admit the following expansion

$$\lambda_{2,\delta}(p) = \lambda_* + \frac{1}{\gamma_*}\sqrt{\delta^2 t_*^2 + \theta_*^2 (p - p_*)^2} + O(p - p_*)^2 + O(\delta^2),$$
$$\lambda_{1,\delta}(p) = \lambda_* - \frac{1}{\gamma_*}\sqrt{\delta^2 t_*^2 + \theta_*^2 (p - p_*)^2} + O(p - p_*)^2 + O(\delta^2).$$

Thus $I_\delta := [\lambda_{1,\delta}(0), \lambda_{2,\delta}(0)]$ is a band gap. The asymptotics of $u_{1,\delta}(p)$ and $u_{2,\delta}(p)$ can be derived correspondingly.



Main result on the existence of interface eigenvalue

Theorem

Assume that $t_* \neq 0$. Then for δ sufficiently small, there exists a unique point spectrum of $\tilde{\mathscr{L}}_\delta : H^2_b(\tilde{\Omega}_\delta) \subset L^2(\tilde{\Omega}_\delta) \to L^2(\tilde{\Omega}_\delta)$ in the interval

$$I_\delta \equiv (\lambda_* - c\delta|\beta_*|, \lambda_* + c\delta|\beta_*|),$$

where $0 < c < 1$ is a constant and $\beta_* = \frac{t_*}{\gamma_*}$.

(c)

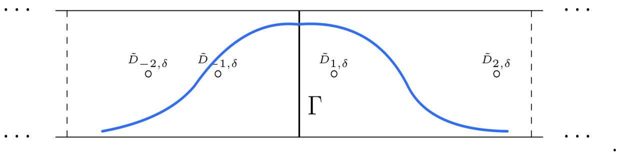


Figure: Interface state in the perturbed structure.

Proof of the existence of interface eigenvalue: representation of interface mode

If $u(x;\lambda)$ is an interface mode, then

$$u(x;\lambda)=\begin{cases}u^R(x), & x_1>0,\\ u^L(x), & x_1<0,\end{cases}\quad \text{with}\quad \begin{cases}u(0,x_2,\lambda)=u^L(0^-,x_2;\lambda)=u^R(0^+,x_2;\lambda),\\ \dfrac{\partial u}{\partial x_1}(0,x_2;\lambda)=\dfrac{\partial u^L}{\partial x_1}(0^-,x_2;\lambda)=\dfrac{\partial u^R}{\partial x_1}(0^+,x_2;\lambda).\end{cases}$$

We can prove the following boundary integral representation:

$$u^R(x;\lambda)=2\int_\Gamma G_\delta(x,y;\lambda)\phi(y)d\sigma(y),$$
$$u^L(x;\lambda)=-2\int_\Gamma G_{-\delta}(-x,y;\lambda)\phi(y)d\sigma(y),$$

where $\phi=\frac{\partial u}{\partial x_1}(0,x_2;\lambda)\in H^{-\frac{1}{2}}(\Gamma)$ and G_δ is the Green function in the perturbed waveguide:

$$\begin{cases}(\Delta_x+\lambda)G_\delta(x,y;\lambda)=\delta(x-y), & x,y\in\Omega_\delta,\\ G_\delta(x,y;\lambda)=0, & x\in\bigcup_{n\in\mathbf{Z}}\partial D_{n,\delta},\\ \dfrac{\partial}{\partial x_2}G_\delta(x,y;\lambda)=0, & x\in\Gamma_1\bigcup\Gamma_2,\\ G_\delta(\cdot,y;\lambda)\ \text{satisfies the radiation condition.}\end{cases}$$

Proof of the existence of interface eigenvalue: matching of boundary conditions

The condition $u^L(0^-, x_2; \lambda) = u^R(0^+, x_2; \lambda)$ leads to the following boundary integral equation:

$$\tilde{\mathscr{G}}_\delta(\lambda)\phi := \Big(\mathscr{G}_\delta(\lambda) + \mathscr{G}_{-\delta}(\lambda)\Big)\phi = 0,$$

where

$$\mathscr{G}_\delta(\lambda) : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \quad \varphi(y) \mapsto \int_\Gamma G_\delta(x,y;\lambda)\varphi(y)d\sigma(y),$$
$$\mathscr{G}_{-\delta}(\lambda) : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \quad \varphi(y) \mapsto \int_\Gamma G_{-\delta}(x,y;\lambda)\varphi(y)d\sigma(y).$$

One can show that the interface eigenvalue problem is equivalent to the following characteristic value problem

$$\tilde{\mathscr{G}}_\delta(\lambda)\phi = 0, \quad \phi \in H^{-\frac{1}{2}}(\Gamma).$$

Spectral expansion of the Green function

We denote $\{(\lambda_{j,\delta}(p), u_{j,\delta}(x;p))\}_{j\geq 1}$ the Bloch eigenpairs of the perturbed periodic structure Ω_δ for each $p \in B = [0, 2\pi]$. Then $G_{\pm\delta}$ attains the following spectral representation for λ in band gaps:

$$G_{\pm\delta}(x,y;\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=1}^{+\infty} \frac{u_{n,\pm\delta}(x;p)\overline{u_{n,\pm\delta}(y;p)}}{\lambda - \lambda_{n,\pm\delta}(p)} dp.$$

The expansion for the layer potentials $\mathscr{G}_{\pm\delta}(\lambda)$ follows accordingly.

Key step: limiting behaviour of the operator $\tilde{\mathscr{G}}_\delta$

Proposition

Let I_δ be the band gap and $\tilde{I} := \{h \in \mathbb{R} : \lambda_* + \delta \cdot h \in I_\delta\}$. Then the following convergence holds uniformly for $h \in \tilde{I}$:

$$\lim_{\delta \to 0} \left\| \tilde{\mathscr{G}}_\delta(\lambda_* + \delta \cdot h) - \left(2T_0 + \beta(h)\mathbb{P}\right) \right\|_{\mathscr{B}(H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma))} = 0.$$

where

$$T_0 = \frac{1}{2\pi} \sum_{n \geq 3} \int_0^{2\pi} \frac{\langle \cdot, \overline{u_n(x;p)} \rangle}{\lambda_* - \lambda_n(p)} u_n(x;p)dp$$

$$+ \frac{1}{2\pi} \sum_{n=1,2} p.v. \int_{[0,2\pi]} \frac{\langle \cdot, \overline{u_n(x;p)} \rangle}{\lambda_* - \lambda_n(p)} u_n(x;p)dp,$$

$$\beta(h) = -\frac{1}{\beta_* \alpha_*} \frac{h}{\sqrt{1 - (\frac{h}{\beta_*})^2}}, \quad \mathbb{P} = \langle \cdot, \overline{u_1(x;\pi)} \rangle u_1(x;\pi) + \langle \cdot, \overline{u_2(x;p)} \rangle u_2(x;\pi).$$

Moreover, T_0 is a Fredholm operator with index 0 and with a Kernel of dimension 1.

Remark: T_0 is associated with evanescent waves while $\mathbb{P}$ propagating waves.

Proof of the existence of interface eigenvalue

The following two characteristic value problem have the same number of solutions:

$$\tilde{\mathscr{G}}_\delta(\lambda_* + \delta\cdot h)\phi = 0 \quad VS \quad \Big(2T_0 + \beta(h)\mathbb{P}\Big)\phi = 0.$$

Proposition

The operator $2T_0 + \beta(h)\mathbb{P}$ has a unique simple characteristic value $h = 0$ and invertible for $h \neq 0$.

Thus, the original characteristic problem has a unique solution $\lambda^\star = \lambda_* + \delta\cdot h^\star$ for some $h^* \in \tilde{I}$. Let $\phi^\star \in H^{-\frac{1}{2}}(\Gamma)$ be the associated root function, then the interface mode is given by

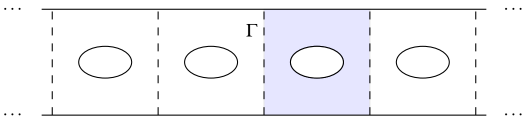
$$u^\star = \begin{cases} \int_\Gamma G_\delta((x_1,x_2),y;\lambda)\phi^\star(y)d\sigma(y), & x_1 \geq 0, \\ -\int_\Gamma G_{-\delta}((-x_1,x_2),y;\lambda)\phi^\star(y)d\sigma(y), & x_1 < 0. \end{cases}$$

Interface modes in a waveguide without band gap opening

Reference: Jiayu Qiu and H. Zhang, On the bifurcation of a Dirac point in a photonic waveguide without band gap opening, preprint.

Setup of the model

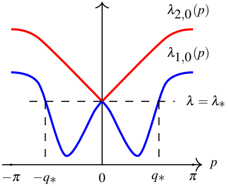
We start with the following unperturbed periodic waveguide Ω with holes and which is filled with material with refractive index $n_{\varepsilon=0}(\boldsymbol{x})$.



The wave propagation can be modelled by the following equations

$$\begin{cases} (\mathscr{L}_{\varepsilon=0} - \lambda)u = 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases}$$

where $\mathscr{L}_{\varepsilon=0} = -\frac{1}{n^2_{\varepsilon=0}(\boldsymbol{x})}\Delta$. Assume $\mathscr{L}_{\varepsilon=0}$ has the band structure as below, i.e. a Dirac point at $p = 0$ which is at the lower end of the second band, but inside the first band



For simplicity, we assume reflection symmetry: $[\mathscr{P}, \mathscr{L}_{\varepsilon=0}] = 0$, $\mathscr{P}[f](x_1, x_2) := f(-x_1, x_2)$.

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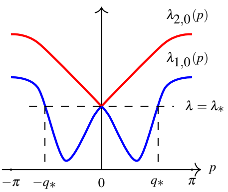
We consider the perturbed refractive index $n_\varepsilon(x)$ and assume that

$$\Big(\frac{\partial \mathscr{L}_\varepsilon}{\partial \varepsilon}\big|_{\varepsilon=0} v_n(\cdot;p=0), v_m(\cdot;p=0)\Big)_{n,m=1,2} = \begin{pmatrix} 0 & t_* \\ t_* & 0 \end{pmatrix}, \quad t_* \neq 0.$$

$v_1(\boldsymbol{x};p), v_2(\boldsymbol{x};p)$: first two branches of the analytic Bloch eigenfunctions. Assume reflection symmetry: $[\mathscr{P}, \mathscr{L}_\varepsilon] = 0$. Then $\mathscr{L}_\varepsilon$ $(0 < |\varepsilon| \ll 1)$ has the band structure as in Figure (b).



(a) Unperturbed band (b) Perturbed band

Remark: The band gap is not opened by the small perturbation. The second band is lifted up from the Dirac point.

Interface modes and resonant modes

We consider the joint structure that is modelled by the operator

$$(\mathscr{L}_\varepsilon^\star u)(x_1,x_2) := \begin{cases} (\mathscr{L}_\varepsilon u)(x_1,x_2), & x_1 > 0, \\ (\mathscr{L}_{-\varepsilon} u)(x_1,x_2), & x_1 < 0. \end{cases}$$

Let $\mathscr{I}_\varepsilon = \{\lambda \in \mathbf{C} : |\lambda - \lambda_*| < |t_*|\varepsilon\}$. The main result is:

Theorem

For $0 < |\varepsilon| \ll 1$, $\mathscr{L}_\varepsilon^\star$ has a generalized eigenvalue $\lambda^\star \in \mathscr{I}_\varepsilon \cap \{Im(\lambda) \leq 0\}$ with the eigenfunction $u^\star \in L^2_{loc}(\Omega)$. Moreover,

1. *$\|u^\star\| < \infty$ if and only if $Im(\lambda^\star) = 0$;*
2. *$\|u^\star\|_{L^2(\Omega)} = \infty$ if and only if $u^\star|_\Gamma$ can be coupled to either the right going Bloch mode of $\mathscr{L}_\varepsilon$ at $\lambda^\star$ or the left going Bloch mode of $\mathscr{L}_{-\varepsilon}$ at $\lambda^\star$.*

Sketch of the proof

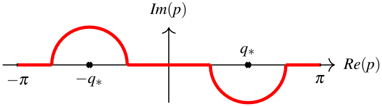
Due to the failure of band gap opening, the Green function (associated with $\mathscr{L}_\varepsilon$, $\varepsilon \neq 0$) is defined via the limiting absorption principle:

$$G_\varepsilon(\boldsymbol{x},\boldsymbol{y};\lambda) := \int_{-\pi}^{\pi} \sum_{n>1} \frac{u_{n,\varepsilon}(\boldsymbol{x};p)\overline{u_{n,\varepsilon}(\boldsymbol{y};p)}}{\lambda - \lambda_{n,\varepsilon}(p)}\,dp + \lim_{\eta\to 0^+} \int_{-\pi}^{\pi} \frac{u_{1,\varepsilon}(\boldsymbol{x};p)\overline{u_{1,\varepsilon}(\boldsymbol{y};p)}}{\lambda + i\eta - \lambda_{1,\varepsilon}(p)}\,dp.$$

Warning: G_ε is not analytic in λ due to the branch cut.
We construct an analytic continuation of G_ε:

$$\tilde{G}_\varepsilon(\boldsymbol{x},\boldsymbol{y};\lambda) := \int_{-\pi}^{\pi} \sum_{n>1} \frac{u_{n,\varepsilon}(\boldsymbol{x};p)\overline{u_{n,\varepsilon}(\boldsymbol{y};p)}}{\lambda - \lambda_{n,\varepsilon}(p)}\,dp + \int_{C_\varepsilon} \frac{u_{1,\varepsilon}(\boldsymbol{x};p)\overline{u_{1,\varepsilon}(\boldsymbol{y};p)}}{\lambda + i\eta - \lambda_{1,\varepsilon}(p)}\,dp.$$

Here C_ε is the integral contour:



Theorem

$\tilde{G}_\varepsilon = G_\varepsilon$ for λ real.

Sketch of the proof

Define

$$\tilde{\mathbb{G}}_{\pm\varepsilon}(\lambda) : \varphi(y) \mapsto \int_\Gamma \tilde{G}_{\pm\varepsilon}(\boldsymbol{x},\boldsymbol{y};\lambda)\varphi(\boldsymbol{y};\lambda)d\sigma(\boldsymbol{y}).$$

Theorem

For $\lambda \in \mathscr{I}_\varepsilon$ and $\varphi \in \tilde{H}^{-\frac{1}{2}}(\Gamma)$, we define $u(x;\lambda) = (\tilde{\mathbb{G}}_\varepsilon(\lambda)\varphi)(x)$ ($x \in \Omega$). Then

$$(\mathscr{L}_\varepsilon - \lambda)u(x) = 0, \quad x \in \Omega,$$

and

$$u(x;\lambda) = u(\mathscr{P}x;\lambda),$$

$$\left(\frac{\partial u}{\partial x_1}\right)\Big|_\Gamma = \frac{\varphi}{2}.$$

Sketch of the proof

We construct a solution to $(\mathscr{L}_\varepsilon^\star - \lambda)u = 0$ with the form

$$u(\boldsymbol{x}) = \begin{cases} \int_\Gamma \tilde{G}_\varepsilon(\boldsymbol{x},\boldsymbol{y};\lambda)\varphi(\boldsymbol{y};\lambda)d\sigma(\boldsymbol{y}), & x_1 > 0, \\ -\int_\Gamma \tilde{G}_{-\varepsilon}(\boldsymbol{x},\boldsymbol{y};\lambda)\varphi(\boldsymbol{y};\lambda)d\sigma(\boldsymbol{y}), & x_1 < 0, \end{cases} \tag{2}$$

This construction requires

$$\left(\tilde{\mathbb{G}}_\varepsilon(\lambda) + \tilde{\mathbb{G}}_{-\varepsilon}(\lambda)\right)\varphi = 0, \tag{3}$$

where

$$\tilde{\mathbb{G}}_{\pm\varepsilon}(\lambda) : \varphi(y) \mapsto \int_\Gamma \tilde{G}_{\pm\varepsilon}(\boldsymbol{x},\boldsymbol{y};\lambda)\varphi(\boldsymbol{y};\lambda)d\sigma(\boldsymbol{y}).$$

We solve (3) to obtain characteristic values which satisfy $\mathrm{Im}(\lambda) \le 0$. Then (2) gives an interface mode (or resonant mode) if and only if $\mathrm{Im}(\lambda) = 0$ (or $\mathrm{Im}(\lambda) < 0$).

Integer Quantum Hall Effect in Square Lattice Photonic structure

Reference: Jiayu Qiu and H. Zhang, A Mathematical Theory of Integer Quantum Hall Effect in Photonics, preprint.

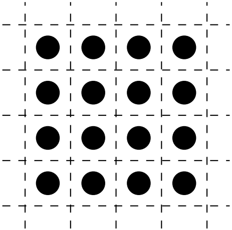
The model setup

We consider the structure is proposed in the first experimental realization of QHE in photonics by Wang et.al. 09: 2D square lattice of magneto-optical particles.

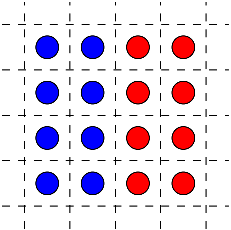
Without applied magnetic field, time-harmonic TE polarized EM wave propagation can be modeled by the following operator

$$\mathscr{L}^A = -div(A\nabla)$$

where $A(\boldsymbol{x}) = \left(1 + c \cdot \sum_{n_1,n_2\in\mathbf{Z}} \chi_{D_{n_1,n_2}}(\boldsymbol{x})\right) \cdot I_{2\times 2}$.



(c) Unperturbed structure (dielectric rods embedded in the air)



(d) Perturbed structure (different magnetic fields at two sides)

Absence of Dirac cones

Assume that $\mathscr{L}^A$ is invariant under $C_{4v}-$point group. The representation theory of C_{4v} group indicates the existence of degenerate points in the band structure of $\sigma(\mathscr{L}^A)$.

Theorem

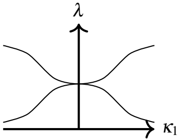
Liner degenerate points (i.e. Dirac points) can't appear at $\boldsymbol{\kappa} = (0,0),(0,\pi)$ *or* (π,π).

We assume

(1) A quadratic degenerate point at M-point. More precisely, the first two bands of $\mathscr{L}^A$ touch quadratically at $(\boldsymbol{\kappa}_* = (\pi,\pi), \lambda_*)$, where the Bloch modes u_1, u_2 satisfy the following relations ($\mathscr{R} : \frac{\pi}{4}-$rotation, $\mathscr{F} : x_1-$reflection)

$$\mathscr{R}u_1 = iu_2, \quad \mathscr{R}u_2 = iu_1, \quad \mathscr{F}u_1 = u_1, \quad \mathscr{F}u_2 = -u_2.$$

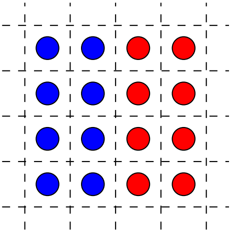
(2) Spectral no-fold condition at $\lambda = \lambda_*$. More precisely, λ_* is at the upper end of the first band and lower end of the second band.

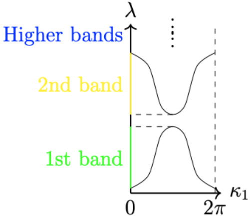


Band gap opening and phase transition

We consider the perturbed operator $\mathscr{L}^{A\pm\delta\cdot B}$ where $B(\boldsymbol{x}) = b(\boldsymbol{x})\cdot\sigma_2$ with $b(\boldsymbol{x}) = \sum_{n_1,n_2\in\mathbf{Z}} \chi_{D_{n_1,n_2}}(\boldsymbol{x})$ and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. This models the propagation of TE waves in the lattice structure under an applied uniform magnetic field (to the gyromagnetic particles).

Remark: $\mathscr{L}^{B}$ breaks the $\mathscr{F}-$symmetry (as well as the time-reversal symmetry), which opens a band gap near $(\boldsymbol{\kappa}_*, \lambda_*)$.





(c)

We obtain by dedicated 2nd order perturbation arguments:

Theorem

For $|\delta|, |\kappa_1 - \pi| \ll 1$*, and* $\boldsymbol{\kappa} = \boldsymbol{\kappa}_* + (\kappa_1 - \pi)\boldsymbol{e}_1$*, the first two branches of Bloch eigenpairs of* $\mathscr{L}^{A \pm \delta \cdot B}$ *satisfy*

$$\lambda_{1,\delta}(\boldsymbol{\kappa}) = \lambda_{1,-\delta}(\boldsymbol{\kappa}) \sim \lambda_* - \sqrt{\delta^2 t_*^2 + \frac{1}{4}\gamma_*^2 |\kappa_1 - \pi|^4},$$

$$\lambda_{2,\delta}(\boldsymbol{\kappa}) = \lambda_{2,-\delta}(\boldsymbol{\kappa}) \sim \lambda_* + \sqrt{\delta^2 t_*^2 + \frac{1}{4}\gamma_*^2 |\kappa_1 - \pi|^4}.$$

where $\gamma_*, t_* \in \mathbf{R}$*. Moreover,*

$$v_{1,\delta}(x;\boldsymbol{\kappa}) \sim v_1(x;\boldsymbol{\kappa}_*) + t_* f(\kappa_1 - \pi;\delta) \cdot v_2(x;\boldsymbol{\kappa}_*) + (\kappa_1 - \pi)(\partial \kappa_1) v_1(x;\boldsymbol{\kappa}_*) + r_1(\boldsymbol{x}),$$

$$v_{2,\delta}(x;\boldsymbol{\kappa}) \sim -t_* f(\kappa_1 - \pi;\delta) \cdot v_1(x;\boldsymbol{\kappa}_*) + v_2(x;\boldsymbol{\kappa}_*) + (\kappa_1 - \pi)(\partial \kappa_1) v_2(x;\boldsymbol{\kappa}_*) + r_2(\boldsymbol{x}),$$

where $f(p;\delta) = \dfrac{\delta}{\frac{1}{2}\gamma_* p^2 + \sqrt{\frac{1}{4}\gamma_*^2 p^4 + t_*^2 \delta^2}}$, $r_1, r_2 = o(1)$.

Interface modes for $\kappa_{\parallel} = \pi$

We consider the joint structure that is modelled by the following operator

$$\mathscr{L}^{inter} = \begin{cases} \mathscr{L}^{A+\delta\cdot B}, & x_1 > 0, \\ \mathscr{L}^{A-\delta\cdot B}, & x_1 < 0. \end{cases}$$

Theorem

Along the e_2 interface, for $\kappa_{\parallel} = \pi$, there exists exactly two eigenvalues $\lambda_n^{\star}$ $(n = 1,2)$ of $\mathscr{L}^{inter}$ in the band gap, whose eigenfunctions decays exponentially away from the interface.

Remark: One can prove there exist exactly two eigenvalues $\lambda_n^{\star}(\kappa_2)$ $(n = 1,2)$ of $\mathscr{L}^{inter}\Big|_{L^2_{\kappa_2}(\Omega)}$ inside the band gap for $|\kappa_2 - \pi| \ll 1$ by a standard perturbation argument. One can also calculate the slope $(\lambda_n^{\star})'(\pi)$ of the two dispersion curves for the interface modes.

Sketch of proof

We can show that the in-gap eigenvalues are equivalent to the characteristic values of the following the integral operator

$$\left(\mathbb{T}^{\delta}(\lambda)+\mathbb{T}^{-\delta}(\lambda)\right)\begin{pmatrix}\phi\\ \varphi\end{pmatrix}=0, \tag{4}$$

where the entries of $\mathbb{T}^{\delta}$ are layer potential operators associated with the Green function G^{δ}

$$\mathbb{T}^{\delta}\in\mathscr{B}(H^{\frac{1}{2}}(\Gamma)\times\tilde{H}^{-\frac{1}{2}}(\Gamma)),\quad \mathbb{T}^{\delta}\begin{pmatrix}\phi\\ \varphi\end{pmatrix}:=\begin{pmatrix}-\mathscr{K}(\lambda;G^{\delta}) & \mathscr{S}(\lambda;G^{\delta})\\ -\mathscr{N}(\lambda;G^{\delta}) & \mathscr{K}^{*}(\lambda;G^{\delta})\end{pmatrix}\begin{pmatrix}\phi\\ \varphi\end{pmatrix}.$$

Technicality: $\mathbb{T}^{\delta}$ blows up at different rates on different subspaces as $\delta\to 0$. We thus consider the following normalized equations:

$$\left(\mathbb{M}^{\delta}(\lambda)+\mathbb{M}^{-\delta}(\lambda)\right)\begin{pmatrix}\Psi\\ \Phi^{(1)}\\ \Phi^{(1)}\end{pmatrix}=0,\quad \mathbb{M}^{\delta}(\lambda):=\begin{pmatrix}\mathbb{Q}\mathbb{T}^{\delta}(\lambda)\mathbb{Q} & \delta^{\frac{1}{4}}\mathbb{Q}\mathbb{T}^{\delta}(\lambda)\Pi_1 & \delta^{-\frac{1}{12}}\mathbb{Q}\mathbb{T}^{\delta}(\lambda)\Pi_2\\ \delta^{\frac{1}{4}}\Pi_2\mathbb{T}^{\delta}(\lambda)\mathbb{Q} & \delta^{\frac{1}{2}}\Pi_2\mathbb{T}^{\delta}(\lambda)\Pi_1 & \delta^{\frac{1}{6}}\Pi_2\mathbb{T}^{\delta}(\lambda)\Pi_2\\ \delta^{-\frac{1}{12}}\Pi_1\mathbb{T}^{\delta}(\lambda)\mathbb{Q} & \delta^{\frac{1}{6}}\Pi_1\mathbb{T}^{\delta}(\lambda)\Pi_1 & \delta^{-\frac{1}{6}}\Pi_1\mathbb{T}^{\delta}(\lambda)\Pi_2\end{pmatrix}.$$

Here Π_1: projection to $(\partial\kappa_1 v_n)(\boldsymbol{x};\boldsymbol{\kappa}_*)$, Π_2: projection to $v_n(\boldsymbol{x};\boldsymbol{\kappa}_*)$ $(n=1,2)$.

Theorem

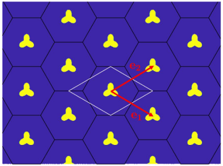
(4) has two characteristic values, which gives two in-gap interface eigenvalues.

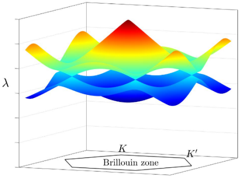
Interface modes in honeycomb structure

Reference: Jiayu Qiu, Junshan Lin, Wei Li and H. Zhang, Mathematical Theory for Interface Modes in Honeycomb Topological Wave Insulator with Broken Reflection Symmetry, preprint.

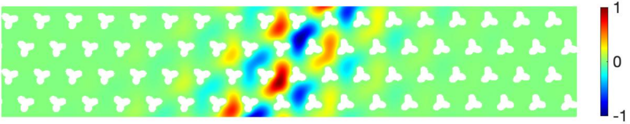
Interface modes in honeycomb structure

Idea: We start with a honeycomb structure with a Dirac point. This can be achieved by assuming lattice symmetry + $\frac{2\pi}{3}$ rotation symmetry+ reflection symmetry w.r.t x-axis.





We then perturb the structure differently on the two sides of an interface to create an interface mode.



Existence of Dirac points in honeycomb structure

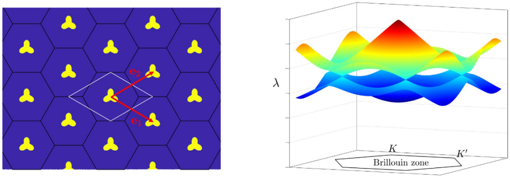
Let $C_z = \{l_1 e_1 + l_2 e_2 : l_1, l_2 \in (-1/2, 1/2)\}$ be the fundamental cell, $D(\eta)$ be an impenetrable inclusion with size η located at the center of C_z. Assume $D(\eta)$ is invariant under the $\frac{2\pi}{3}$-rotation and the horizontal reflection.

Theorem

For $\eta \ll 1$, there exists a Dirac point at (K, λ_*) in the band structure of the honeycomb lattice. The dispersion surface near (K, λ_*) takes the form

$$(\lambda - \lambda_*)^2 = m_*^2 |p - K|^2 + O(|p-K|^3), \quad m_* \in \mathbb{R}, \quad m_* \geq 0,$$

where $m_* = \frac{2}{3}(1 + O(\eta))$.



Theorem

Assume $t_* > 0$. The following dispersion relations hold for p near K and λ near λ_*:

$$\lambda_{1,\pm\varepsilon}(p) = \lambda_* - \frac{1}{|\gamma_*|}\sqrt{\varepsilon^2 t_*^2 + m_*^2|\gamma_*|^2|p-K|^2}\,\big(1+O(\varepsilon,|p-K|)\big),$$

$$\lambda_{2,\pm\varepsilon}(p) = \lambda_* + \frac{1}{|\gamma_*|}\sqrt{\varepsilon^2 t_*^2 + m_*^2|\gamma_*|^2|p-K|^2}\,\big(1+O(\varepsilon,|p-K|)\big).$$

Remark: m_*, γ_* are constants which depend on the unperturbed structure. t_* depends on the perturbation.

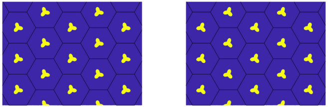


Figure 1.2: The two perturbed honeycomb lattices by rotating the obstacles counter-clockwisely and clockwisely.

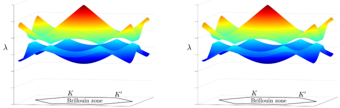


Figure 1.3: The band structure of the two perturbed honeycomb lattices in Figure 1.2.

Theorem

Assume that $t_* \neq 0$, and η be an arbitrary constant in $(0,1)$. For sufficiently small positive ε, there exist a unique interface mode with $k_{\parallel}^{*,a} = 2\pi$, with the corresponding eigenvalues $\lambda_{\pm} \in (\lambda_* - \eta \frac{t_*}{\gamma_*}\varepsilon, \lambda_* + \eta \frac{t_*}{\gamma_*}\varepsilon)$. In addition, both edge modes decay exponentially as $|x \cdot e_1| \to \infty$.

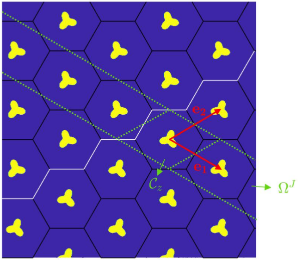


Figure 2.2: Join wave insulator with a zigzag interface.

Interface modes along an armchair interface

Theorem

Assume that $t_* \neq 0$, and $\eta \in (0,1)$. For sufficiently small positive ε, there exist exactly two edge modes with $k_{\parallel}^{*,a} = 2\pi$, with the corresponding eigenvalues $\lambda_{\pm} \in (\lambda_* - \eta \frac{t_*}{\gamma_*}\varepsilon, \lambda_* + \eta \frac{t_*}{\gamma_*}\varepsilon)$. In addition, both edge modes decay exponentially as $|x \cdot e_1^a| \to \infty$.

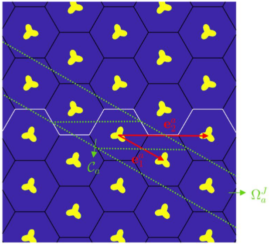


Figure 2.4: Join wave insulator with an armchair interface.

Summary

1. Existence of in-gap interface eigenvalues bifurcated from Dirac points for various models in photonics/phononics;
2. Existence of in-gap interface eigenvalues bifurcated from quadratic degenerate points in 2D square lattice of photonic structures;
3. Compared to the results in the domain-wall models, the new approach can overcome the difficulties of discontinues coefficients and sharp interfaces.
4. New ideas are need to establish bulk-interface correspondence type results in the non-perturbative regime!

Thank you for your attention!