Graph Theory Prelim 2021

Reminders:

• I need to see you *and* your workspace.
• Speaker *on* and mic *off*.
• This exam is completely closed book.
• Have your phone in reach just to use for submission.
• *Question?* Raise your hand to ask to chat.
• *Done?* Raise your hand to ask to start submission.

**Please be sure to number your answers.*

A few definitions:

• Given any simple graph G on n vertices, K_n can be decomposed into G and the n-vertex graph $\overline{G}$, called the *complement of* G. If G and $\overline{G}$ are isomorphic, then G is said to be *self-complementary.*

• The *connectivity of a graph* G, denoted $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex.

• Given a graph G and a set $X \subseteq V(G)$, the *graph induced by* X, denoted $G[X]$, is the subgraph of G with vertex set X and whose edge set comprises precisely those edges of G with both ends in X.

1. In this question all graphs are simple.

   (a) Let G be a graph on n vertices that is self-complementary. Prove that $n \equiv 0, 1 \pmod 4$.

   (b) Prove that if $n \equiv 0 \pmod 4$, then there exists a self-complementary graph on n vertices. *(Hint: Divide the vertices into four equal sets, and try to mimic the structure of P_4)*

2. Let $G_0 = (A, B)$ be a bipartite graph with a perfect matching, where $|A| = |B| = n/2$. Let G be obtained from G_0 by adding edges so that $G[A]$ and $G[B]$ are both connected graphs.

   (a) Prove that $\kappa(G) \geq 1 + \min\{\kappa(G[A]), \kappa(G[B])\}$.
   (Hint: Show $G - X$ is connected $\forall$ X of appropriate size).

   (b) Construct a family of examples to show that if "perfect matching" is replaced by "$\delta(G_0) \geq 1$", the right-hand-side of (a) can be arbitrarily larger than $\kappa(G)$.

3. (a) State Hall's Theorem for bipartite graphs.

   (b) Use Hall's Theorem to show that regular bipartite graphs have perfect matchings. *(Hint: First show shores have equal size)*

   (c) Use (b) to prove that cubic bipartite graphs have nz 3-flows.

4. It is a fact that all planar graphs have a vertex of degree at most 5. Use this to prove (a) and (b) by induction.

   (a) Every loopless planar graph is 6-colourable.

   (b) Every loopless planar graph is 5-colourable.

   (c) State Tutte's 5-Flow Conjecture and discuss what results are known towards it.