Graph Theory Prelim, 8/20/2022

1) a) Carefully state A. J. Hoffman's circulation theorem. Include all relevant definitions. (Do **not** include a proof!!)

b) In class, we gave conditions on a capacitated digraph that guaranteed that if it had a feasible circulation, then it had an integer valued one. State these conditions.

c) In class, using b), we proved that if f is a circulation in the digraph D, then there is a circulation g in D which satisfies $g(e) \in \{ \lfloor f(e) \rfloor , \lceil f(e) \rceil \}$, for all arcs e in D. Using this, or otherwise, prove the following :

Let k be a positive integer, let a(1) , a(2), … , a(k) be real numbers summing to the integer s. Then there are integers b(1) , b(2) , … , b(k) , also summing to s, with $b(j) \in \{\lfloor a(j) \rfloor , \lceil a(j) \rceil\}$ for all $1 \leq j \leq k$.

2) a) Carefully state Phillip Hall's theorem on matchings in bipartite graphs.

b) Using a), or otherwise, prove that the edges of a regular bipartite graph can be partitioned into perfect matchings.

c) Give a counter example to show that b) is false if the word "bipartite" is removed.

3) **Theorem. (Dirac, 1952)** *Let G be a simple graph with $n \geq 3$ vertices. If $\delta(G) \geq n/2$, then G is Hamiltonian.*

Suppose you wanted to prove Dirac's Theorem by induction on n and that, in the general case/ inductive step, you removed one vertex v from the graph G. If G-v is Hamiltonian, would that imply that G is Hamiltonian? Explain. Would you be able to apply induction to G-v to conclude that it is Hamiltonian? Explain. *(Note: No inductive proofs of Dirac's Theorem are actually known, although perhaps a nice one is just waiting to be found!)*

4) Here is my definition of a *condensation* graph H(C, K) of the simple graph G(V, E). Suppose that f : V **onto** C be a proper coloring of G, that is if $uv \in E$, then $f(u) \neq f(v)$. We declare $a, b \in C$ to be adjacent in H, that is, $ab \in K$, if there is an edge $uv \in E$ with f(u) = a, f(v) = b.

a) Prove that G itself is one of its condensations.

b) Show that a condensation graph of G with the fewest possible number of vertices is a complete graph.