STAT 7600/7610 Mathematical Statistics Preliminary Exam

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Statistics Group, Department of Mathematics and Statistics, Auburn University

Name:________________________

1. It is a closed-book in-class exam. You are allowed to have formula sheets of two pages and double-sided.
2. A calculator is allowed.
3. The proctor will provide as many blank sheets of paper as you need.
4. Show your work to receive full credit. Highlight your final answer.
5. Turn in your the exam paper (the three typeset pages handed to you) along with your work-sheets stabled to the back.
6. Planned Time: 240 minutes (8:00 am-12:00(noon)).
7. **Five** problems will be graded. Problems 1 and 2 are mandatory. Then, you **must** select **three problems** from Problems 3–7 to submit and grade. The rest problems will not be graded. Indicate your selections in the table.

1	2	3	4	5	6	7	Total

The density, mean, and variance of selected common distributions.

- Normal (μ, σ^2)

$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) \quad E(X) = \mu \quad \text{var}(X) = \sigma^2$$

- Gamma(α, β) (shape-rate parametrization)

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad E(X) = \alpha/\beta \quad \text{var}(X) = \alpha/\beta^2$$

  If $X_1 \sim \text{Gamma}(\alpha_1, \beta)$, $X_2 \sim \text{Gamma}(\alpha_2, \beta)$, and X_1 is independent of X_2, then

$$\beta(X_1 + X_2) \sim \text{Gamma}(\alpha_1 + \alpha_2, 1).$$

- χ^2_p

$$f(x) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2} \quad E(X) = p \quad \text{var}(X) = 2p$$

- Beta(α, β)

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad E(X) = \frac{\alpha}{\alpha+\beta} \quad \text{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

- Poission(λ)

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad E(X) = \lambda \quad \text{var}(X) = \lambda$$

Throughout, the asymptotics whenever mentioned is in terms of the sample size approaching infinity.

Mandatory questions

Problem 1: Let $X_1, \ldots, X_n$ be i.i.d. random variables with density function

$$f(x|\theta) = \frac{\theta}{2^\theta} x^{\theta-1}, \quad 0 < x < 2, \quad \theta > 0.$$

(a) Find the maximum likelihood estimator (MLE) of θ.

(b) Find the maximum likelihood estimator (MLE)of $\eta = 1/\theta$. State any property/result you are using when answering this.

(c) Determine if the MLE of η is biased. If it is, find the bias.

Problem 2: Under the same setting as Problem 1.

(a) Find the Fisher Information $I(\theta)$ for this distribution.

(b) Find the asymptotic distribution of the MLE $\hat{\theta}$ as $n \to \infty$.

(c) Applying the delta-method, find the asymptotic distribution of the MLE $\hat{\eta}$ as $n \to \infty$.

(d) Construct an approximate 95% confidence interval for θ.

(e) Construct the Wald test for testing the hypothesis $H_0 : \theta = \theta_0$ v.s. $H_a : \theta \neq \theta_0$. State the test statistic and the rejection region at the asymptotic level 5%.

Choose three from the five

Problem 3: Let $X_1, \ldots, X_n$ be i.i.d. random variables with the density function

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \quad x > 0, \quad \mu > 0,\ \lambda > 0.$$

(a) Suppose that λ is known and μ is the parameter. Show that the given family of distributions belongs to the exponential family. You need to identify all components in the pdf of an exponential family distribution.

(b) As in (a), find a minimal sufficient statistic for μ.

(c) If both λ and μ are unknown parameters, find a minimal sufficient statistic for μ and λ.

(d) Find $E(\sum_{j=1}^n X_j)$ and $E(\sum_{j=1}^n 1/X_j)$.

Problem 4: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the probability density function

$$f(x \mid \mu) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\log x - \mu)^2}{2}\right), \quad x > 0, \quad \mu \in \mathbb{R}.$$

(a) Derive the likelihood ratio test (LRT) for testing $H_0 : \mu = \mu_0$ vs. $H_a : \mu \neq \mu_0$.

(b) Determine the asymptotic distribution of the test statistic under H_0. Compute the critical value for the LRT and determine the rejection region at the asymptotic level α.

(c) Invert the test to obtain a $(1 - \alpha) \times 100\%$ approximate confidence interval for μ.

Problem 5: Let $(X_1, \ldots, X_n)$ be iid from a distribution with density

$$f(x \mid \theta) = \frac{\theta^3}{2} x^2 e^{-\theta x}, \quad x > 0, \quad \theta > 0.$$

(a) Find a minimal sufficient statistic T_n for the family of distributions with $\theta > 0$.

(b) Show that the distribution family has monotone likelihood ratio in T_n.

(c) Find the uniformly most powerful test for $H_0 : \theta \leq \theta_0$ versus $H_a : \theta > \theta_0$ at the significance level α. Express the rejection region and find the critical value (in a closed form).

Problem 6: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a Gamma distribution with shape parameter α and rate parameter β, i.e.,

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0, \quad \alpha > 0, \beta > 0.$$

(a) Find a pivotal quantity for constructing a confidence interval for β assuming α is known.

(b) Construct a confidence interval for β using the pivotal quantity found in part (a).

(c) Derive the likelihood ratio test (LRT) for testing $H_0 : \beta = \beta_0$ v.s. $H_a : \beta \neq \beta_0$ assuming α is known. State how the critical value can be determined at a given significance level α.

(d) Construct a confidence interval for β by inverting the acceptance region of the LRT derived in part (c).

Problem 7: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a Poisson distribution with parameter λ, i.e.,

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots, \quad \lambda > 0.$$

In the following, use the asymptotic significance level α and the asymptotic confidence level $1-\alpha$.

(a) Derive the Wald test for testing $H_0 : \lambda = \lambda_0$ vs. $H_a : \lambda \neq \lambda_0$. Clearly state the test statistic, the rejection region and the critical value.

(b) Show that the score test is equivalent to the Wald test.

(c) Consider the simple hypothesis $H_0 : \lambda = \lambda_1$ v.s. $H_a : \lambda = \lambda_2$ where λ_2 and λ_1 are fixed constants. What is the most powerful test of level α for the hypothesis? Clearly state the test statistic and state how to determine the critical value.

(d) Is there a uniformly most powerful (UMP) test for any $\lambda_2 > \lambda_1$? Explain.